Analytical Solution of Two-Impulse Transfer Between Coplanar Elliptical Orbits

CONTROLA LAB SEMINAR
- DAY 3 -

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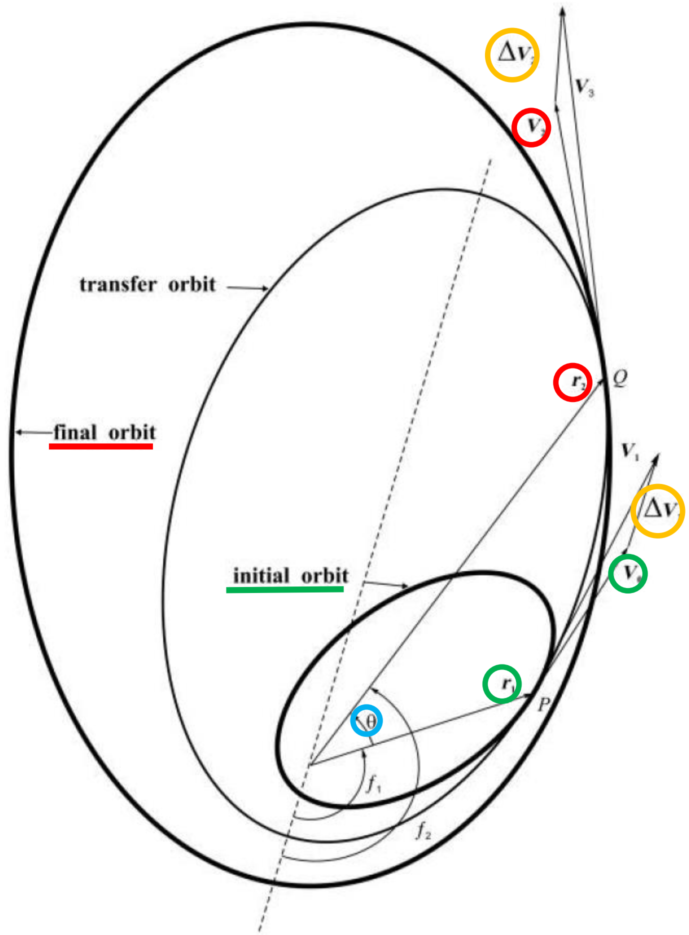
PROBLEM STATEMENT(1/6)

A. Coplanar Two-Impulse Transfer(1/2)



- Given Values
  - Initial Orbit: $\boldsymbol{r}_1,\ \boldsymbol{v}_0$
  - Final Orbit: $\boldsymbol{r}_2,\ \boldsymbol{v}_3$
  - Fixed transfer angle: θ

- Performance Index

$$J = \Delta v_\Sigma = \Delta v_1 + \Delta v_2$$

Where,

$$\Delta v_1 = \|\boldsymbol{v}_1 - \boldsymbol{v}_0\| = \sqrt{(v_{1r} - v_{0r})^2 + (v_{1\theta} - v_{0\theta})^2}$$

$$\Delta v_2 = \|\boldsymbol{v}_3 - \boldsymbol{v}_2\| = \sqrt{(v_{3r} - v_{2r})^2 + (v_{3\theta} - v_{2\theta})^2}$$

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PROBLEM STATEMENT(2/6)

A. Coplanar Two-Impulse Transfer(2/2)

- For the sake of convenience,

$$\boldsymbol{v}_1 = \begin{bmatrix} v_{1r} \\ v_{1\theta} \end{bmatrix} = \sqrt{\frac{\mu}{p}} \begin{bmatrix} e\sin f_1 \\ 1+e\cos f_1 \end{bmatrix} = \frac{\sqrt{\mu}}{p_H} \begin{bmatrix} e_H \sqrt{p}\cot\frac{\theta}{2} + \left(\frac{p_H}{\sqrt{p}} - \sqrt{p}\right)\tan\frac{\theta}{2} \\ (1+e_H)\sqrt{p} \end{bmatrix}$$

$$\boldsymbol{v}_2 = \begin{bmatrix} v_{2r} \\ v_{2\theta} \end{bmatrix} = \sqrt{\frac{\mu}{p}} \begin{bmatrix} e\sin f_2 \\ 1+e\cos f_2 \end{bmatrix} = \frac{\sqrt{\mu}}{p_H} \begin{bmatrix} e_H \sqrt{p}\cot\frac{\theta}{2} - \left(\frac{p_H}{\sqrt{p}} - \sqrt{p}\right)\tan\frac{\theta}{2} \\ (1-e_H)\sqrt{p} \end{bmatrix}$$

Where,

$$p_H = \frac{2r_1 r_2}{r_1 + r_2}, \quad e_H = \frac{r_2 - r_1}{r_2 + r_1}$$

From above Eq.

$$r_1 = \frac{p}{1+e\cos f_1} = \frac{p_H}{1+e_H}, \qquad r_2 = \frac{p}{1+e\cos f_2} = \frac{p_H}{1-e_H}$$

As a result, the problem of the optimal transfer between coplanar orbits is reduced to the search for the semi-latus rectum(p) that minimizes performance index(J) when boundary condition and fixed transfer angle are specified.

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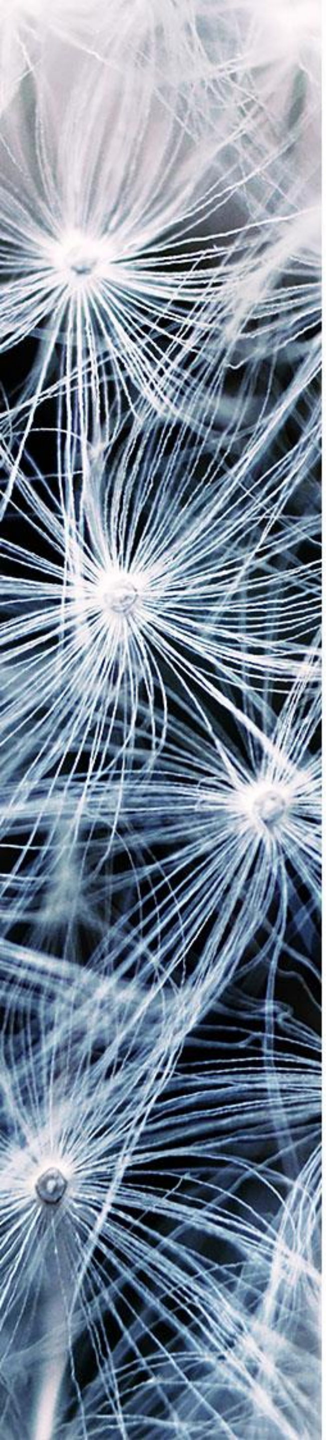


PROBLEM STATEMENT(3/6)

B. Necessary Condition for Optimality

- **A necessary condition for the optimality**

$$\frac{\partial J}{\partial\sqrt{p}}=\frac{\partial\Delta v_1}{\partial\boldsymbol{v}_1}\frac{\partial\boldsymbol{v}_1}{\partial\sqrt{p}}+\frac{\partial\Delta v_2}{\partial\boldsymbol{v}_2}\frac{\partial\boldsymbol{v}_2}{\partial\sqrt{p}}=0$$

Where,

$$\frac{\partial\Delta v_1}{\partial\boldsymbol{v}_1}=\begin{bmatrix}\sin\varphi_1\\ \cos\varphi_1\end{bmatrix}$$

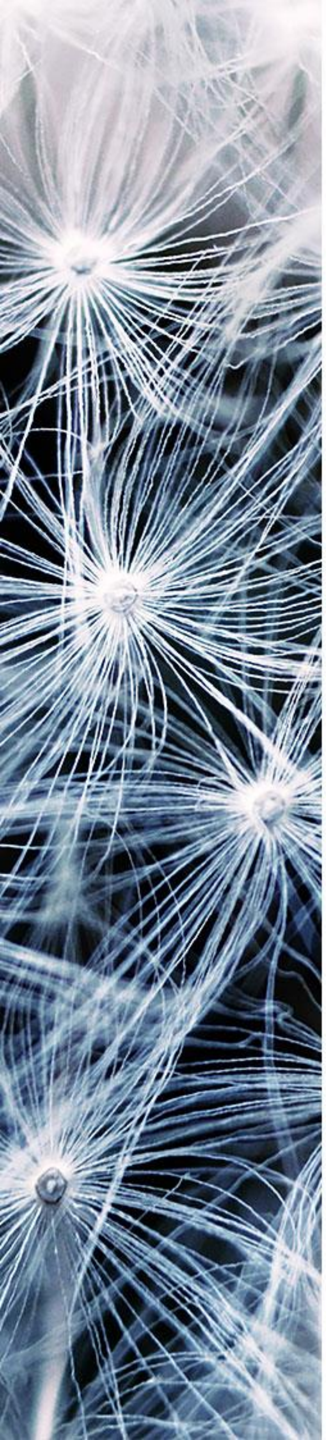
$$\frac{\partial\Delta v_2}{\partial\boldsymbol{v}_2}=-\begin{bmatrix}\sin\varphi_2\\ \cos\varphi_2\end{bmatrix}$$

$$\frac{\partial\boldsymbol{v}_1}{\partial\sqrt{p}}=\frac{\sqrt{\mu}}{p_H}\begin{bmatrix}e_H\cot\frac{\theta}{2}-\left(\frac{p_H}{p}+1\right)\tan\frac{\theta}{2}\\ 1+e_H\end{bmatrix}$$

$$\frac{\partial\boldsymbol{v}_2}{\partial\sqrt{p}}=\frac{\sqrt{\mu}}{p_H}\begin{bmatrix}e_H\cot\frac{\theta}{2}+\left(\frac{p_H}{p}+1\right)\tan\frac{\theta}{2}\\ 1-e_H\end{bmatrix}$$

Therefore,

$$\frac{\partial J}{\partial\sqrt{p}}=\frac{\sqrt{\mu}}{p_H}\left[\left(\sin\varphi_1-\sin\varphi_2\right)e_H\cot\frac{\theta}{2}-\left(\sin\varphi_1+\sin\varphi_2\right)\left(\frac{p_H}{p}+1\right)\tan\frac{\theta}{2}+\left(1+e_H\right)\cos\varphi_1-\left(1-e_H\right)\cos\varphi_2\right]=0$$



PROBLEM STATEMENT(4/6)

C. Characteristics of the Primer Vector and Necessary Condition for Optimality(1/2)

- The Primer vectors are

$$\tilde{\boldsymbol{p}}_1 = \begin{bmatrix} \tilde{p}_{1r} \\ \tilde{p}_{1\theta} \end{bmatrix} = \begin{bmatrix} \sin\varphi_1 \\ \cos\varphi_1 \end{bmatrix} = \begin{bmatrix} A\cos f_1 + Be\sin f_1 \\ -A\sin f_1 + B(1+e\cos f_1) + \dfrac{D - A\sin f_1}{1+e\cos f_1} \end{bmatrix}$$

$$\tilde{\boldsymbol{p}}_2 = \begin{bmatrix} \tilde{p}_{2r} \\ \tilde{p}_{2\theta} \end{bmatrix} = \begin{bmatrix} \sin\varphi_1 \\ \cos\varphi_1 \end{bmatrix} = \begin{bmatrix} A\cos f_2 + Be\sin f_2 \\ -A\sin f_2 + B(1+e\cos f_2) + \dfrac{D - A\sin f_2}{1+e\cos f_2} \end{bmatrix}$$

Where A, B, D are Lawden's Constants

It is necessary to enforce that

$$\|\tilde{\boldsymbol{p}}_1\| = \|\tilde{\boldsymbol{p}}_2\| = 1$$

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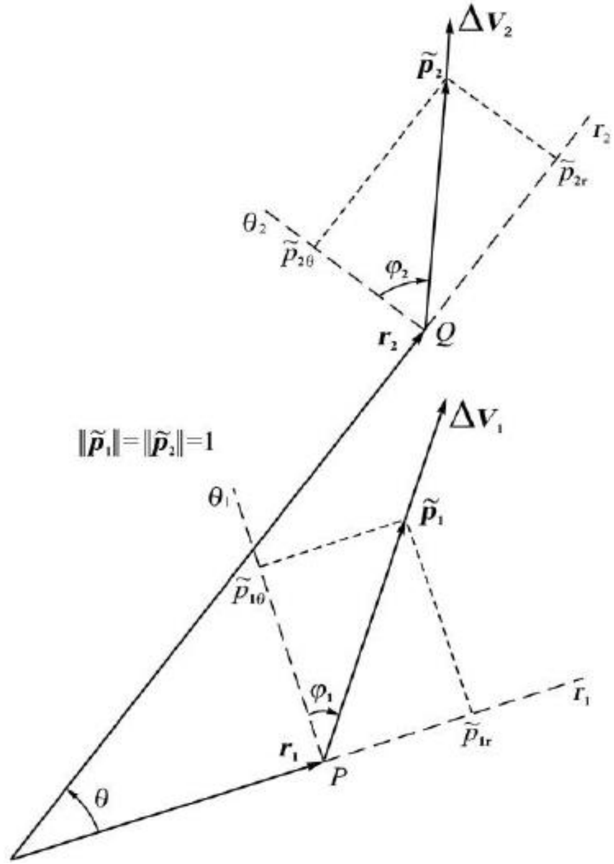
PROBLEM STATEMENT(5/6)

C. Characteristics of the Primer Vector and Necessary Condition for Optimality(2/2)



- The Primer vectors are written as follows

$$\tilde{\boldsymbol{p}}_1 = \sqrt{\frac{p}{\mu}}\begin{bmatrix} Bv_{1r} + A\frac{1}{e}\left(\frac{p}{p_H}(1+e_H)-1\right) \\ Bv_{1\theta} - A\frac{1}{e}\left(1+\frac{p_H}{p}\frac{1}{1+e_H}\right)v_{1r} + D\sqrt{\frac{\mu}{p^3}}\frac{p_H}{1+e_H} \end{bmatrix}$$

$$\tilde{\boldsymbol{p}}_2 = \sqrt{\frac{p}{\mu}}\begin{bmatrix} Bv_{2r} + A\frac{1}{e}\left(\frac{p}{p_H}(1-e_H)-1\right) \\ Bv_{2\theta} - A\frac{1}{e}\left(1+\frac{p_H}{p}\frac{1}{1-e_H}\right)v_{2r} + D\sqrt{\frac{\mu}{p^3}}\frac{p_H}{1+e_H} \end{bmatrix}$$

- From the Primer vector continuity for impulsive thrust,

$$\tan\varphi_1 = \frac{v_{1r}-v_{0r}}{v_{1\theta}-v_{0\theta}}, \qquad \tan\varphi_2 = \frac{v_{3r}-v_{3r}}{v_{3\theta}-v_{2\theta}}$$

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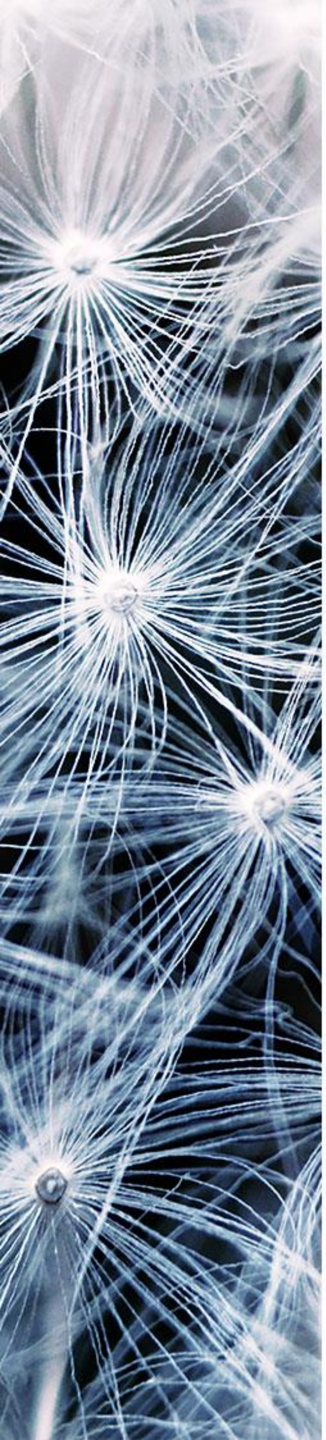

PROBLEM STATEMENT(6/6)

D. Problem Statement

- If varying the transfer angle and semi-latus rectum, one reach the global minimum delta velocity.

★Finally, The solution achieved as a result defines an optimum transfer orbit at an optimum orientation of the initial and final orbits with the fixed points set on them.

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SOLUTION OF
THE OPTIMIZATION PROBLEM

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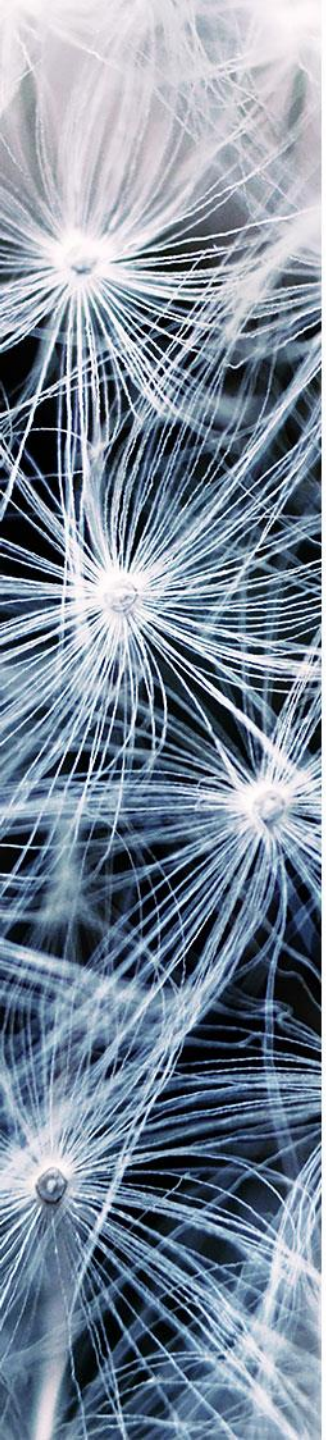


SOLUTION OF THE OPTIMIZATION PROBLEM(1/8)

A. Necessary Conditions for Optimality

- Necessary conditions to find such $p = p^{opt}$ and $\theta = \theta^{opt}$

$$\frac{\partial J}{\partial \sqrt{p}} = 0, \qquad \frac{\partial J}{\partial \theta} = 0$$

Where second necessary condition is

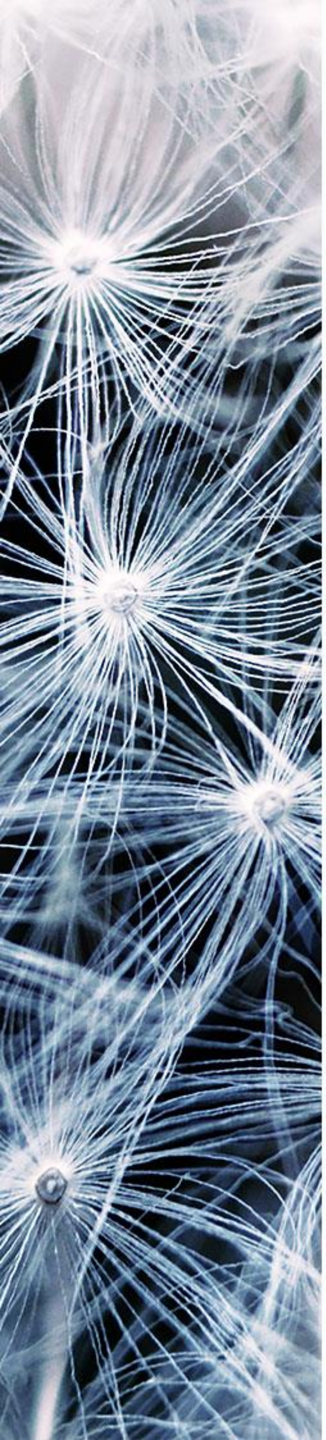
$$\frac{\partial J}{\partial \theta} = \frac{\partial \Delta v_1}{\partial \mathbf{v}_1}\frac{\partial \mathbf{v}_1}{\partial \theta} + \frac{\partial \Delta v_2}{\partial \mathbf{v}_2}\frac{\partial \mathbf{v}_2}{\partial \theta} = \frac{v_{1r}\sin\varphi_1 - v_{2r}\sin\varphi_2}{\sin\theta} = 0$$

Then,

$$\frac{\partial J}{\partial \theta} = \sqrt{\frac{\mu}{p}}\,\frac{e\sin f_1\left(A\cos f_2 + Be\sin f_2\right) - e\sin f_2\left(A\cos f_1 + Be\sin f_1\right)}{\sin\theta}$$

$$= \sqrt{\frac{\mu}{p}}eA = 0$$

$$\therefore A = 0$$



SOLUTION OF THE OPTIMIZATION PROBLEM(2/8)

B. Solution for Lawden's Constants(1/2)

- **If A=0,**

$$\tilde{\boldsymbol{p}}_1 = \sqrt{\frac{p}{\mu}}\begin{bmatrix} Bv_{1r} \\ Bv_{1\theta} + D\sqrt{\dfrac{\mu}{p^3}}\dfrac{p_H}{1+e_H} \end{bmatrix}$$

$$\tilde{\boldsymbol{p}}_2 = \sqrt{\frac{p}{\mu}}\begin{bmatrix} Bv_{2r} \\ Bv_{2\theta} + D\sqrt{\dfrac{\mu}{p^3}}\dfrac{p_H}{1-e_H} \end{bmatrix}$$

- **From this Eq.**

$$\|\tilde{\boldsymbol{p}}_1\| = \sqrt{\frac{p}{\mu}\left[\left(Bv_{1r}\right)^2 + \left(Bv_{1\theta} + D\sqrt{\frac{\mu}{p^3}}\frac{p_H}{1+e_H}\right)^2\right]} = 1$$

$$\|\tilde{\boldsymbol{p}}_2\| = \sqrt{\frac{p}{\mu}\left[\left(Bv_{1r}\right)^2 + \left(Bv_{1\theta} + D\sqrt{\frac{\mu}{p^3}}\frac{p_H}{1-e_H}\right)^2\right]} = 1$$

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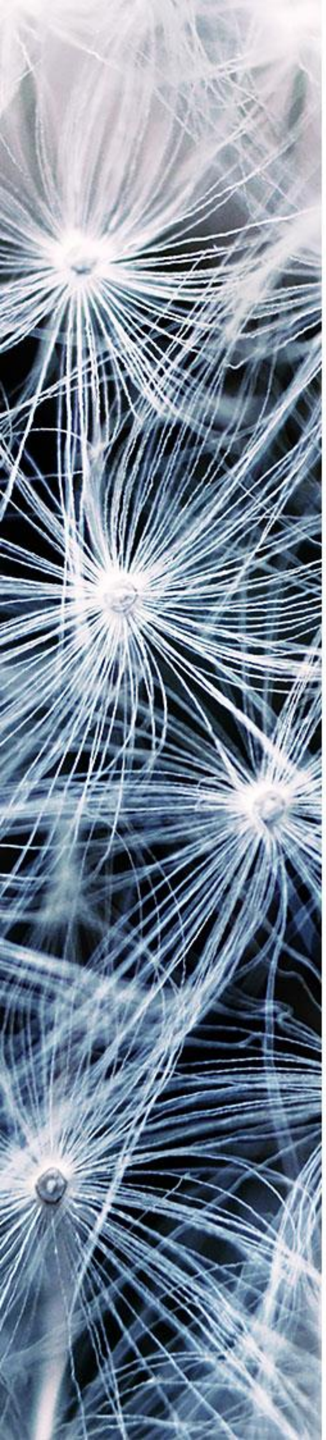

SOLUTION OF THE OPTIMIZATION PROBLEM(3/8)

B. Solution for Lawden's Constants(2/2)

- Therefore,

$$B = \sqrt{\frac{\mu}{p}}\frac{1}{\Delta\tilde{v}}, \quad D = \pm\sqrt{\frac{\mu}{p_H}}\frac{p}{p_H}\left(1-e_H^2\right)\frac{1}{\Delta\tilde{v}}$$

Where,

$$\Delta\tilde{v} = \sqrt{v_{1r}^2 + \left(v_{1\theta} \pm v_{2H\theta}\right)^2} = \sqrt{v_{2r}^2 + \left(v_{2\theta} \pm v_{1H\theta}\right)^2}$$

And

$$v_{1H\theta} = \sqrt{\frac{\mu}{p_H}}\left(1+e_H\right), \quad v_{2H\theta} = \sqrt{\frac{\mu}{p_H}}\left(1-e_H\right)$$

(normal in-plain components of velocities for Hohmann's transfer orbits.)

- Thrust angles are

$$\tan\varphi_1 = \frac{v_{1r}}{v_{1\theta} \pm v_{1H\theta}}, \qquad \tan\varphi_2 = \frac{v_{2r}}{v_{2\theta} \pm v_{1H\theta}}$$

Thus, If substitute to the equation earlier,

$$\tan\varphi_1^{opt} = \frac{v_{0r}}{v_{0\theta} \pm v_{2H\theta}}, \quad \tan\varphi_2^{opt} = \frac{v_{3r}}{v_{3\theta} \pm v_{1H\theta}}$$

SOLUTION OF THE OPTIMIZATION PROBLEM(4/8)

C. Solution of the Transfer Semi-latus Rectum and Transfer Angle(1/2)

- From the necessary conditions

$$\left(\sin\varphi_1^{opt}-\sin\varphi_2^{opt}\right)e_H\cot\frac{\theta}{2}-\left(\sin\varphi_1^{opt}+\sin\varphi_2^{opt}\right)\left(\frac{p_H}{p}+1\right)\tan\frac{\theta}{2}+\left(1+e_H\right)\cos\varphi_1^{opt}-\left(1-e_H\right)\cos\varphi_2^{opt}=0$$

$$\left(\sin\varphi_1^{opt}-\sin\varphi_2^{opt}\right)e_H\cot\frac{\theta}{2}-\left(\sin\varphi_1^{opt}+\sin\varphi_2^{opt}\right)\left(\frac{p_H}{p}-1\right)\tan\frac{\theta}{2}=0$$

- Finally,

$$p^{opt}=p_H\left[\frac{\left(1+e_H\right)\cos\varphi_1^{opt}-\left(1-e_H\right)\cos\varphi_2^{opt}}{\left(1+e_H\right)\cos\varphi_2^{opt}-\left(1-e_H\right)\cos\varphi_1^{opt}}\right]$$

$$\tan\frac{\theta^{opt}}{2}=\frac{1}{2}\frac{\left(1+e_H\right)\cos\varphi_1^{opt}-\left(1-e_H\right)\cos\varphi_2^{opt}}{\sin\varphi_1^{opt}+\sin\varphi_2^{opt}}$$

SOLUTION OF THE OPTIMIZATION PROBLEM(5/8)

C. Solution of the Transfer Semi-latus Rectum and Transfer Angle(2/2)

- **Solution for minimum of the total delta velocity**

$$\Delta v_1^{opt} = \Delta\tilde{v} - \Delta\tilde{v}_1$$
$$\Delta v_2^{opt} = \Delta\tilde{v}_2 - \Delta\tilde{v}$$

Where,

$$\Delta\tilde{v}_1 = \sqrt{v_{0r}^2 + \left(v_{0\theta} \pm v_{2H\theta}\right)^2}$$
$$\Delta\tilde{v}_2 = \sqrt{v_{3r}^2 + \left(v_{3\theta} \pm v_{1H\theta}\right)^2}$$
$$\Delta\tilde{v} = 4\sqrt{\frac{\mu}{p_H}} \frac{e_H}{\left(1+e_H\right)\cos\varphi_2^{opt} - \left(1-e_H\right)\cos\varphi_1^{opt}}$$

Therefore,

$$\therefore \Delta v_\Sigma^{opt} = \Delta v_1^{opt} + \Delta v_2^{opt} = \Delta\tilde{v}_2 - \Delta\tilde{v}_1$$

★ **Finally,**

$$\therefore \Delta v_\Sigma^{opt} = \sqrt{v_{3r}^2 + \left(v_{3\theta} + v_{1H\theta}\right)^2} - \sqrt{v_{0r}^2 + \left(v_{0\theta} + v_{2H\theta}\right)^2}$$

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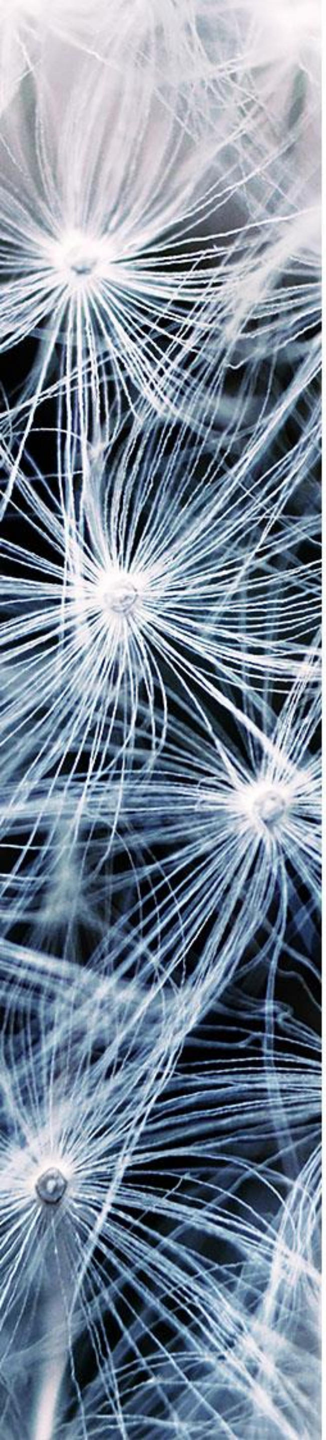

SOLUTION OF THE OPTIMIZATION PROBLEM(7/8)

D. Global Minimum

- The solution needs to be examined according to sufficient conditions of the minimum of function of two variables.

$$\left.\frac{\partial^2 J}{\partial\theta^2}\frac{\partial^2 J}{\partial\left(\sqrt{p}\right)^2}-\left(\frac{\partial^2 J}{\partial\theta\,\partial\sqrt{p}}\right)^2\right|_{\substack{p=p^{opt}\\ \theta=\theta^{opt}}}>0$$

For extreme and

$$\left.\frac{\partial^2 J}{\partial\theta^2}\right|_{\substack{p=p^{opt}\\ \theta=\theta^{opt}}}>0 \quad or \quad \left.\frac{\partial^2 J}{\partial\left(\sqrt{p}\right)^2}\right|_{\substack{p=p^{opt}\\ \theta=\theta^{opt}}}>0$$

After evaluations,

$$\left.\frac{\partial^2 J}{\partial\theta^2}\frac{\partial^2 J}{\partial\left(\sqrt{p}\right)^2}-\left(\frac{\partial^2 J}{\partial\theta\,\partial\sqrt{p}}\right)^2\right|_{\substack{p=p^{opt}\\ \theta=\theta^{opt}}}=\frac{16\mu^2 e_H^2\Delta\tilde{v}_1\Delta\tilde{v}_2}{p_H^3\sin^2\theta^{opt}\Delta\tilde{v}^2\Delta v_1^{opt}\Delta_2^{opt}}>0$$

$$\left.\frac{\partial^2 J}{\partial\theta^2}\right|_{\substack{p=p^{opt}\\ \theta=\theta^{opt}}}=\Delta\tilde{v}\left(\frac{\Delta\tilde{v}_1}{\Delta\tilde{v}_1^{opt}}\cot^2\varphi_1^{opt}+\frac{\Delta\tilde{v}_2}{\Delta\tilde{v}_2^{opt}}\cot^2\varphi_2^{opt}\right)\frac{\sin^2\varphi_1^{opt}\sin^2\varphi_2^{opt}}{\sin^2\theta^{opt}}>0$$

$$\left.\frac{\partial^2 J}{\partial\left(\sqrt{p}\right)^2}\right|_{\substack{p=p^{opt}\\ \theta=\theta^{opt}}}=\frac{\mu}{pp_H}\left[\frac{4e_H}{\Delta\tilde{v}}+\frac{1}{\Delta v_1^{opt}}\left(\frac{e_H\cos\frac{\varphi_1^{opt}-\varphi_2^{opt}}{2}}{\cos\frac{\varphi_1^{opt}+\varphi_2^{opt}}{2}}-\frac{\cos\frac{\varphi_1^{opt}+\varphi_2^{opt}}{2}}{\cos\frac{\varphi_1^{opt}-\varphi_2^{opt}}{2}}\right)\right]^2+\frac{1}{\Delta v_2^{opt}}\left[\left(\frac{e_H\cos\frac{\varphi_1^{opt}-\varphi_2^{opt}}{2}}{\cos\frac{\varphi_1^{opt}+\varphi_2^{opt}}{2}}-\frac{\cos\frac{\varphi_1^{opt}+\varphi_2^{opt}}{2}}{\cos\frac{\varphi_1^{opt}-\varphi_2^{opt}}{2}}\right)\right]>0$$

★Therefore, This solution is a global minimum of total delta velocity

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SOLUTION OF THE OPTIMIZATION PROBLEM(8/8)

E. Comparison with Existing Solution

- From the works of Bender and Battin for $\theta = \pi$

$$\begin{aligned}
\Delta v_{\Sigma}^{\pi} &= \sqrt{\left(v_{0r} + v_{3r}\right)^2 + \left(v_{1H\theta} - v_{0\theta} + v_{3\theta} - v_{2H\theta}\right)^2} \\
&= \sqrt{\Delta\tilde{v}_1^2 + \Delta\tilde{v}_2^2 - 2\cos\left(\varphi_1^{opt} + \varphi_2^{opt}\right)\Delta\tilde{v}_1\Delta\tilde{v}_2} \\
&= \Delta v_{\Sigma}^{opt}\sqrt{1 + 4\frac{\Delta\tilde{v}_1\Delta\tilde{v}_2}{\Delta v_{\Sigma}^{opt\,2}}\sin^2\left(\frac{\varphi_1^{opt}}{2} + \frac{\varphi_2^{opt}}{2}\right)} \\
&\Rightarrow \Delta v_{\Sigma}^{opt} \le \Delta v_{\Sigma}^{\pi}
\end{aligned}$$

★ That is, This solution is more optimal for $\theta = \pi$

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THANK YOU.

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