Quaternions and the rotation of a rigid body

JuneMo Kim

Dept. of Aerospace Engineering

Lab. For Navigation Control, and Applications

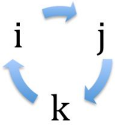
July 25, 2016

content

quaternion

사원수(quaternion)?

1. 복소수를 확장한 다원수이며 회전을 표현할 때 사용
2. 실수 성분 하나와 허수 성분 세 개로 표현 $q = (q_0, q_1, q_2, q_3) = (q_0, \mathbf{q})$
3. 오일러 회전시 발생하는 짐벌락(Gimbal lock) 현상이 발생하지 않고 계산량이 적다는 장점과 직관적이지 않는 단점을 가짐
4. 컴퓨터 애니메이션, 3D게임, 로봇공학 등에 많이 활용



허수의 곱

$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j$$

quaternion

쿼터니언 연산

1) 덧셈

$$q + p = (q_0 + p_0, q_1 + p_1, q_2 + p_2, q_3 + p_3) = (q_0 + p_0, \mathbf{q} + \mathbf{p})$$

2) 뺄셈

$$q - p = (q_0 - p_0, q_1 - p_1, q_2 - p_2, q_3 - p_3) = (q_0 - p_0, \mathbf{q} - \mathbf{p})$$

3) 곱셈

$$q * p = (q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3) + (q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2)i + (q_0 p_2 - q_1 p_3 + q_2 p_0 + q_3 p_1)j + (q_0 p_3 + q_1 p_2 - q_2 p_1 + q_3 p_0)k$$

$$= (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}, q_0 \cdot \mathbf{p} + p_0 \cdot \mathbf{q} + \mathbf{q} \times \mathbf{p})$$

4) 켤레(Conjugate)

$$\bar{q} = (q_0, -q_1, -q_2, -q_3)$$

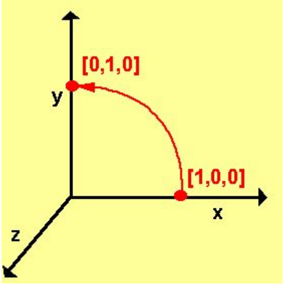
quaternion

5) Rotation

:회전변환을 사원수 곱으로 나타내는 방법

-Example

z축을 기준으로 [1,0,0] $\Longrightarrow$ [0,1,0] $90°(=\frac{\pi}{2})$ 회전



$$q=(cos(\tfrac{\pi}{2}),x_1\cdot sin(\tfrac{\pi}{2}),y_1\cdot sin(\tfrac{\pi}{2}),z_1\cdot sin(\tfrac{\pi}{2}))$$

$$p_1=(0,1,0,0)\quad q=(q_0,\mathbf{u}\cdot sin(\tfrac{\pi}{2}))$$

$$p_2=q*p_1*\tilde{q}$$

$$=(0.7071+0.7071k)*(i)*(0.7071\text{-}0.7071k)$$

$$=(0.7071i+0.7071j)*(0.7071\ \text{-}0.7071k)$$

$$=0.5i+0.5j+0.5j-0.5i=j(0,1,0)$$

Hamiltonian

Hamiltonian

:최소 작용의 원리(Least action principle)를 수식적으로 표현한 식

$$\begin{cases} L = \frac{1}{2}m\dot{x}^2 - v(x) & \text{Lagrangian} \\ H = P\dot{x} - L & \text{Hamiltonian} \end{cases}$$

$$H = (m\dot{x})\dot{x} - (\frac{1}{2}m\dot{x}^2 - v(x))$$

$$\therefore H = \frac{1}{2}m\dot{x}^2 + v(x) = T(kinetic\ energy) + V(potential\ energy)$$

Rotation of a rigid body

-Kinetic energy of a rotating rigid body

$$T = \frac{1}{2}\omega . \mathbb{I}\, \omega$$

-$\mathbb{I}$(tensor of principal moments of inertia)

Ex)

$$(\lambda, \mu) \longrightarrow \lambda \bigotimes \mu : \mathbb{R}^4 x \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$\lambda \bigotimes \mu = (\lambda_0 \mu_0, \lambda_1 \mu_1, \lambda_2 \mu_2, \lambda_3 \mu_3)$$

-Time derivative of the Quaternions

$$\dot{q} = \frac{1}{2}\omega q \Longrightarrow \omega = 2\tilde{q}\dot{q}$$

Rotation of a rigid body

$$T = 2(\tilde{q}\dot{q}) \cdot (\mathbf{I} \bigotimes \tilde{q}\dot{q})$$

Proposition

The function $F(a, b) = (ab) \cdot (\mathbf{I} \bigotimes ab)$, $(a, b{=}\text{quaternions})$

$$\frac{\partial F}{\partial b} = 2\tilde{a}(\mathbf{I} \otimes ab)$$

with this result, Q(conjugate moment of the quaternion q)

$$Q = \frac{\partial T}{\partial \dot{q}} = 4q(\mathbf{I} \otimes \tilde{q}\dot{q})$$

$$\therefore \dot{q} = \frac{1}{4}q(\mathbf{I}^{-1} \otimes \tilde{q}Q)$$

Rotation of a rigid body

Thus, the kinetic energy is

$$T = \frac{1}{8}(\mathbf{I}^{-1}\otimes\tilde{q}Q)\cdot(\tilde{q}Q) \quad or \quad T = \frac{1}{8}(\mathbf{I}^{-1}\otimes\tilde{Q}q)\cdot(\tilde{Q}q) \ \ (\because \widetilde{pq} = \tilde{q}\tilde{p})$$

the equations of motion are

$$\begin{cases} \dot{Q} = -\frac{\partial H}{\partial q} = -\frac{\partial(T+U)}{\partial q} = -\frac{1}{4}Q(\mathbf{I}^{-1}\otimes\tilde{Q}q) - \frac{\partial U(q)}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial Q} = \frac{\partial(T+U)}{\partial Q} = \frac{1}{4}q(\mathbf{I}^{-1}\otimes\tilde{q}Q), \ \ (\because \frac{\partial U(q)}{\partial(Q)} = 0) \end{cases}$$

Applications to gravity gradients

1.Heavy rigid body

-the center of the mass of the body frame

$$\mathbf{x}^c = x_1^c \mathbf{b}_1 + x_2^c \mathbf{b}_2 + x_3^c \mathbf{b}_3$$

therefore, the force in the body frame

$$\mathbf{F} = \mathbf{F}|_B = \tilde{q}\mathbf{F}|_S q = (q_0 - q_1 i - q_2 j - q_3 k)(-Mgs_3)(q_0 \ q_1 i \ q_2 j \ q_3 k)$$
$$= -Mg \cdot (q_3 - q_2 i \ q_1 j \ q_0 k) \ (q_0 \ q_1 i \ q_2 j \ q_3 k)$$
$$\therefore -Mg \cdot (0, 2(q_1 q_3 - q_0 q_2), \ 2(q_0 q_1 + q_2 q_3), \ q_0^2 - q_1^2 - q_2^2 + q_3^2)$$

$\Longrightarrow$ the potential is

$$U = \mathbf{x}^c \cdot \mathbf{F}$$

Applications to gravity gradients

2.Rigid body in a Keplerian orbit

the potential is

$$U = \frac{3GM}{2r^3}(\tilde{q}(p\mathbf{s_1}\tilde{\mathbf{p}})q) \cdot \mathbf{I} \otimes (\tilde{q}(p\mathbf{s_1}\tilde{\mathbf{p}})q)$$

-quaternion p : the rotation from the space frame **S** to the orbital frame(i, ω, Ω: orbit parameters)

$$p = \begin{pmatrix} p_0 = cos\frac{i}{2}cos\frac{\Omega+(\omega+f)}{2} & p_1 = sin\frac{i}{2}cos\frac{\Omega-(\omega+f)}{2} \\ p_2 = sin\frac{i}{2}sin\frac{\Omega-(\omega+f)}{2} & p_3 = cos\frac{i}{2}sin\frac{\Omega+(\omega+f)}{2} \end{pmatrix}$$

Applications to gravity gradients

1)Example(Heavy rigid body)

-initial conditions

$q(t_0) = (0.5, -1/\sqrt{2}, 0, 0.5)$ $\quad$ $\mathbf{x^c} = (\mathbf{1}, \mathbf{0}, \mathbf{0}), \mathbf{Mg} = 0.5$

$Q(t_0) = (0.3, -0.848525, 0.141421, -1.5)$ $\quad$ $(I_1, I_2, I_3) = (1.25, 1.0, 0.75)$

$$U = \mathbf{x^c} \cdot \mathbf{F} = -2Mg(q_1 q_3 - q_0 q_2)$$

$$\Longrightarrow \frac{\partial U}{\partial q} = -2MG \begin{bmatrix} \frac{\partial U}{\partial q_0} \\ \frac{\partial U}{\partial q_1} \\ \frac{\partial U}{\partial q_2} \\ \frac{\partial U}{\partial q_3} \end{bmatrix} = -2MG \begin{bmatrix} -q_2 \\ q_3 \\ -q_0 \\ q_1 \end{bmatrix}$$

Applications to gravity gradients

1)Example(Heavy rigid body)

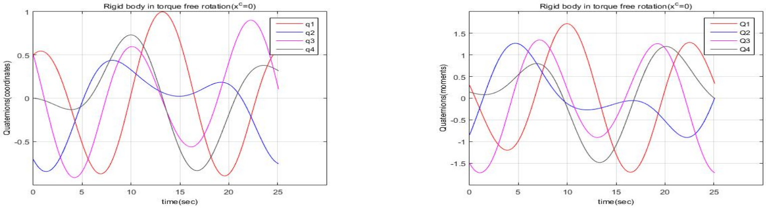


Figure: Rigid body in torque free rotation($x^c = 0$)

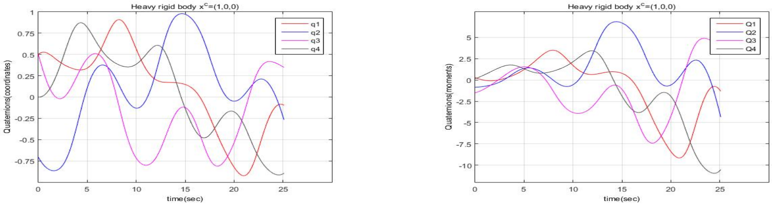


Figure: Heavy rigid body($x^c = (1, 0, 0)$)

Applications to gravity gradients

2)Example(Rigid body in a Keplerian orbit)

-elliptic orbit with initial conditions

$$a = 7000km \ e = 0.2 \ i = \frac{\pi}{6} \ \Omega = \frac{\pi}{4} \ \omega = \frac{\pi}{12} \ f(t_0) = 0$$

-moments of inertia of the satellite

$$I_1 = 400kg/km^2 \ I_2 = 500kg/km^2 \ I_3 = 600kg/km^2$$

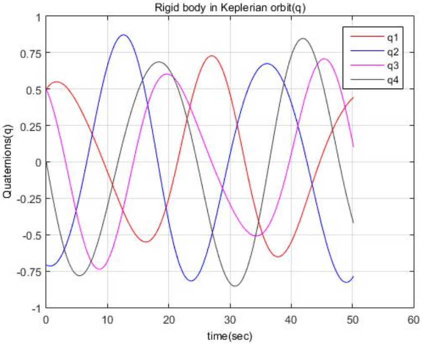
-potential

$$U = \frac{3GM}{2r^3}(\tilde{q}(p\mathbf{s}_1\tilde{\mathbf{p}})q) \cdot \mathbf{I} \otimes (\tilde{q}(p\mathbf{s}_1\tilde{\mathbf{p}})q)$$

$$\therefore \frac{\partial U}{\partial q} = \frac{3GM}{r^3}(\widetilde{\tilde{q}(p\mathbf{s}_1\tilde{\mathbf{p}})}) \cdot \mathbf{I} \otimes (\tilde{q}(p\mathbf{s}_1\tilde{\mathbf{p}})q)$$

Applications to gravity gradients

2)Example(Rigid body in a Keplerian orbit)



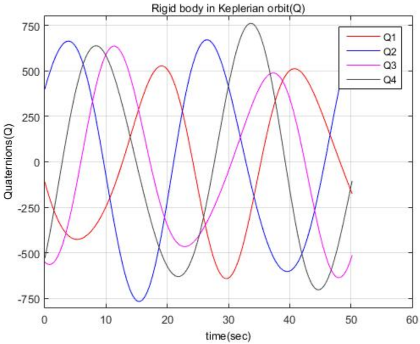


Figure: Rigid body in Keplerian orbit

Future plan

- Hamiltonian
- Dual Quaternion

Thank you for your attention!