SPACECRAFT INTERCEPTOR

CONTROLA LAB SEMINAR
- Day 2 -

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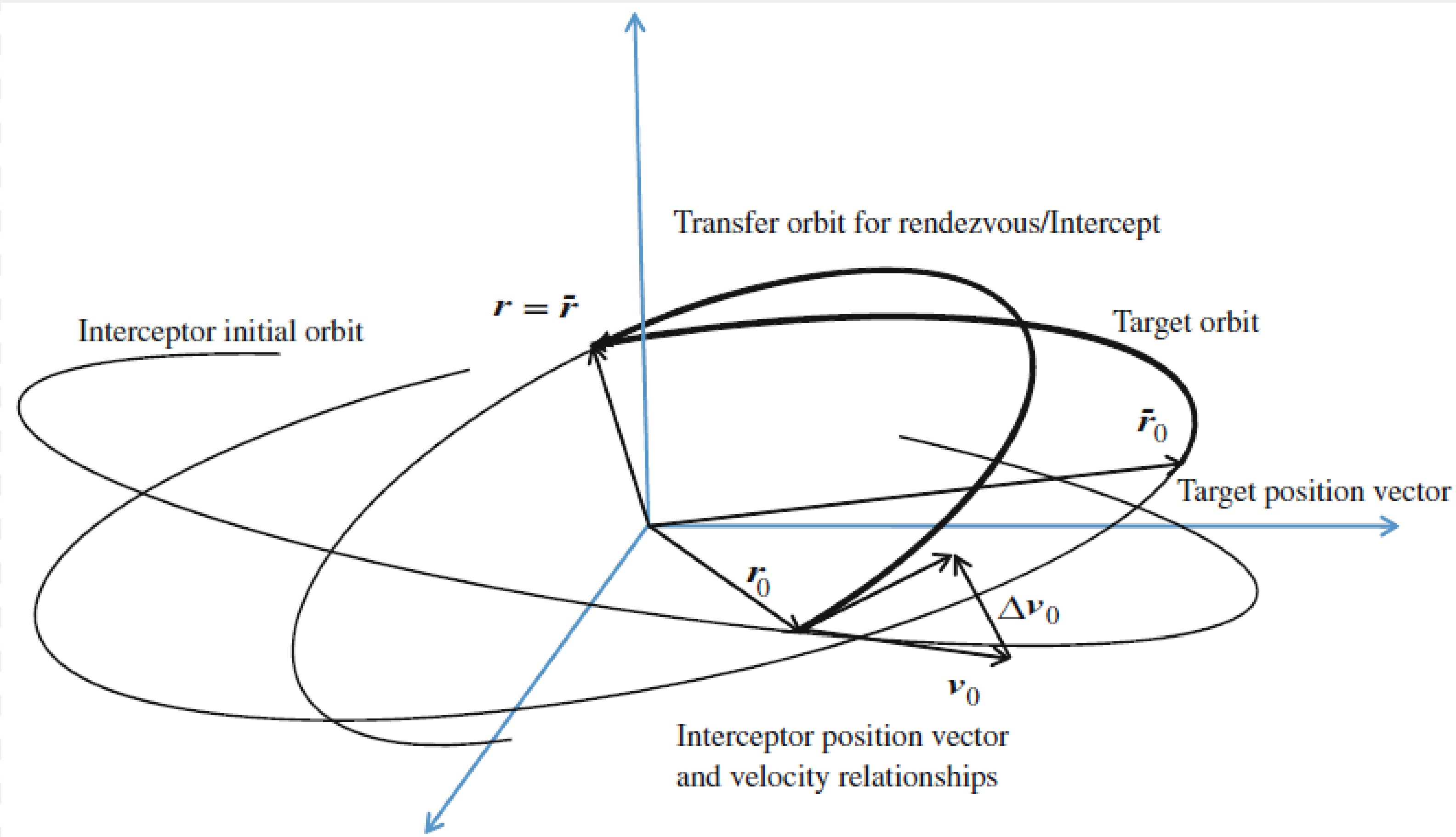
CONTROLA

CONTENTS

INTRODUCTION

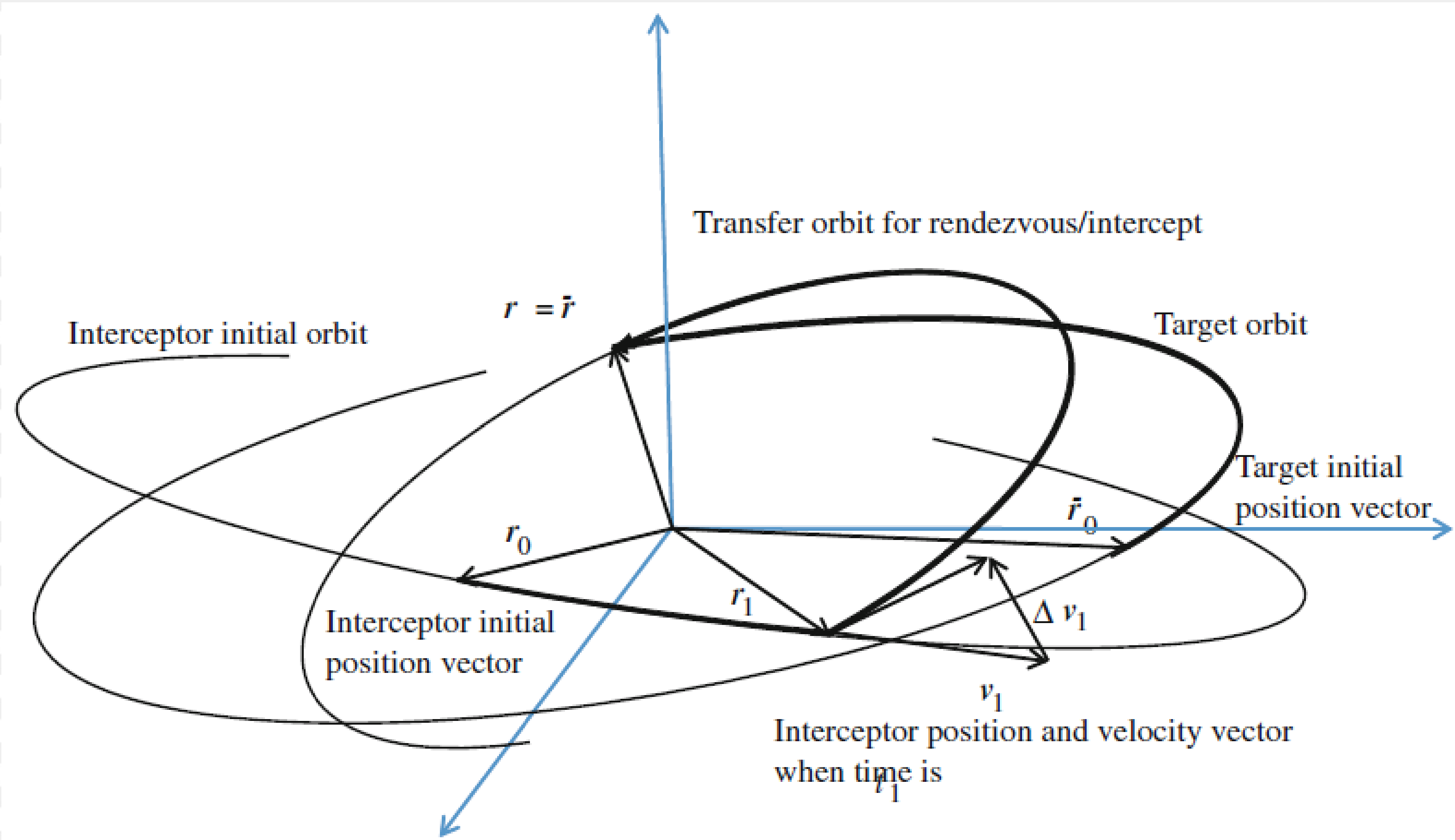
MISSION #1

Minimum Velocity Change



MISSION #2

Minimum Velocity Change and Wait Time



REVIEW

- Kepler's Equation -

REVIEW

Kepler's Equation(1/2)

- Kepler's equation

$$M = n\left(t - T\right) = E - e\sin E$$

- Universal Variable

$$\dot{x} = \frac{\sqrt{\mu}}{r}$$

- The corresponding expression for both for time-of-flight and position

$$\sqrt{\mu}t = a\left(x - \sqrt{a}\sin\frac{x}{\sqrt{a}}\right) + a\frac{\mathbf{r}_0^T\mathbf{v}_0}{\sqrt{\mu}}\left(1 - \cos\frac{x}{\sqrt{a}}\right) + r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right)$$

$$r = a + a\left[\frac{\mathbf{r}_0^T\mathbf{v}_0}{\sqrt{\mu a}}\sin\frac{x}{\sqrt{a}} + \left(\frac{r_0}{a} - 1\right)\cos\frac{x}{\sqrt{a}}\right]$$

REVIEW

Kepler's Equation(2/2)

- f and g solutions

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0$$
$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0$$

- It follows as

$$f = 1 - \frac{a}{r_0}\left(1 - \cos\frac{x}{\sqrt{a}}\right)$$
$$g = t - \frac{a}{\sqrt{\mu}}\left(x - \sqrt{a}\sin\frac{x}{\sqrt{a}}\right)$$
$$\dot{f} = -\frac{\sqrt{\mu a}}{r r_0}\sin\frac{x}{\sqrt{a}}$$
$$\dot{g} = 1 - \frac{a}{r}\left(1 - \cos\frac{x}{\sqrt{a}}\right)$$

REVIEW

- Lagrange Multiplier -

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Lagrange Multiplier(1/2)

- Example 1

$$COST: \; f(x,y) = x^2 + y^2$$
$$CONSTRAINT: \; g(x,y) = -3x + y = 2$$

- Solve

$$f(x) = x^2 + (3x+2)^2$$

- The Necessary Condition

$$\frac{df}{dx} = 10x + 6 = 0$$

- Finally,

$$x^* = -\frac{3}{5}, \quad y^* = \frac{1}{5}$$

REVIEW

Lagrange Multiplier(2/2)

- Example 2

$$FUCTION:\ J(x,y)=x^2+y^2$$
$$CONSTRAINT:\ c(x,y)=2x+y+4=0$$

- Cost Function

$$H=J(x,y)+\lambda c(x,y)$$
$$=x^2+y^2+\lambda(2x+y+4)$$

- The Necessary Condition

$$\frac{\partial H(x,y,\lambda)}{\partial x}=2x+2\lambda=0 \quad / \quad \frac{\partial H(x,y,\lambda)}{\partial y}=2y+\lambda=0 \quad / \quad \frac{\partial H(x,y,\lambda)}{\partial \lambda}=2x+y+4=0$$

- Finally,

$$x^*=-1.6,\quad y^*=-0.8,\quad \lambda^*=1.6$$

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ALGORITHM

- Spacecraft Interceptor Problem #1 -

ALGORITHM

Spacecraft Interceptor Problem #1(1/5)

- Orbit Energy is given by

$$\varepsilon = \frac{v_0^2}{2} - \frac{\mu}{r_0}$$

- Performance Index

$$J = \frac{1}{2}\Delta \mathbf{v}_0^T \Delta \mathbf{v}_0$$

- Constraints

$$\boldsymbol{\eta}\left(\overline{x}, x, \Delta v_0, t\right) = 0$$

$$\overline{\mathbf{r}} - \mathbf{r} = 0$$

Where,

$$\boldsymbol{\eta} = \left[\eta_1 \quad \eta_2\right]^T \in R^2$$

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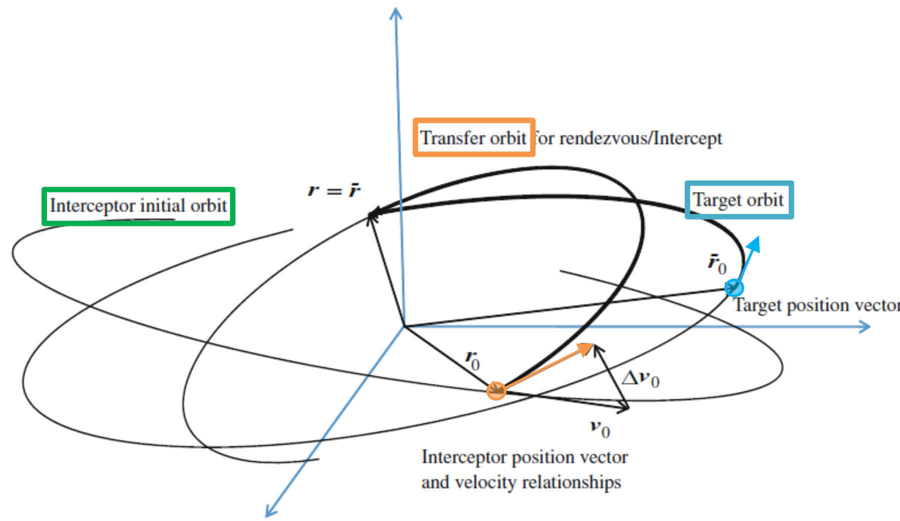
Spacecraft Interceptor Problem #1(2/5)



- Target Orbit

$$\eta_1(\bar{x},t) = \bar{a}\left(\bar{x} - \sqrt{\bar{a}}\sin\frac{\bar{x}}{\sqrt{\bar{a}}}\right) + \bar{a}\frac{\bar{\mathbf{r}}_0^T\bar{\mathbf{v}}_0}{\sqrt{\mu}}\left(1-\cos\frac{\bar{x}}{\sqrt{\bar{a}}}\right) + \bar{r}_0\sqrt{\bar{a}}\sin\left(\frac{\bar{x}}{\sqrt{\bar{a}}}\right) - \sqrt{\mu}t = 0$$

- Transfer Orbit of Interceptor

$$\eta_2(x,t,\Delta v_0) = a\left(x - \sqrt{a}\sin\frac{x}{\sqrt{a}}\right) + a\frac{\mathbf{r}_0^T(\mathbf{v}_0+\Delta\mathbf{v}_0)}{\sqrt{\mu}}\left(1-\cos\frac{x}{\sqrt{a}}\right) + r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right) - \sqrt{\mu}t = 0$$

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Spacecraft Interceptor Problem #1(3/5)

- Cost Function

$$\mathbf{H} = J\left(\Delta \mathbf{v}_0\right) + \boldsymbol{\lambda}^T \boldsymbol{\eta}\left(\bar{x}, x, t, \Delta \mathbf{v}_0\right) + \Phi^T\left(\bar{\mathbf{r}} - \mathbf{r}\right)$$

- The Necessary Condition

$$\frac{\partial H}{\partial \bar{x}} = \lambda_1 \frac{\partial \eta_1}{\partial \bar{x}} + \Phi^T \frac{\partial \bar{\mathbf{r}}}{\partial \bar{x}} = 0$$

$$\frac{\partial H}{\partial x} = \lambda_2 \frac{\partial \eta_2}{\partial x} - \Phi^T \frac{\partial \mathbf{r}}{\partial \bar{x}} = 0$$

$$\frac{\partial H}{\partial t} = -\sqrt{\mu}\left(\lambda_1 + \lambda_2\right) + \Phi^T\left(\frac{\partial \bar{\mathbf{r}}}{\partial t} - \frac{\partial \mathbf{r}}{\partial t}\right) = 0$$

$$\frac{\partial H}{\partial \Delta \mathbf{v}_0} = \frac{\partial J}{\partial \Delta \mathbf{v}_0} + \lambda_2 \frac{\partial \eta_2}{\partial \Delta \mathbf{v}_0} - \Phi^T \frac{\partial \mathbf{r}}{\partial \Delta \mathbf{v}_0} = 0$$

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Spacecraft Interceptor Problem #1(4/5)

- Lagrange Multipliers

$$\lambda_1 = -\Phi^T \frac{1}{\bar{r}} \frac{\partial \bar{\mathbf{r}}}{\partial \bar{x}}$$

$$\lambda_2 = \Phi^T \frac{1}{r} \frac{\partial \mathbf{r}}{\partial x}$$

$$\Phi^T = \frac{\partial J}{\partial \Delta \mathbf{v}_0} \left(\frac{\partial \mathbf{r}}{\partial \Delta \mathbf{v}_0} - \frac{1}{r} \frac{\partial \mathbf{r}}{\partial x} \frac{\partial \eta_2}{\partial \Delta \mathbf{v}_0} \right)^{-1}$$

- The interceptor condition is rewritten as

$$h = \Delta \mathbf{v}_0^T L_0 \left[\left(\frac{\partial \bar{\mathbf{r}}}{\partial \bar{x}} - \frac{\partial \mathbf{r}}{\partial x} \right) + \frac{\bar{r}}{\sqrt{\mu}} \left(\bar{\mathbf{v}}_0 - \left(\mathbf{v}_0 + \Delta \mathbf{v}_0 \right) \right) \right] = 0$$

Where,

$$L_0 = \left(\frac{\partial \mathbf{r}}{\partial \Delta \mathbf{v}_0} - \frac{1}{r} \frac{\partial \mathbf{r}}{\partial x} \frac{\partial \eta_2}{\partial \Delta \mathbf{v}_0} \right)^{-1}$$

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Spacecraft Interceptor Problem #1(5/5)

Finally,

$$\eta_1\left(\bar{x},t\right)=\bar{a}\left(\bar{x}-\sqrt{\bar{a}}\sin\frac{\bar{x}}{\sqrt{\bar{a}}}\right)+\bar{a}\frac{\bar{\mathbf{r}}_0^T\bar{\mathbf{v}}_0}{\sqrt{\mu}}\left(1-\cos\frac{\bar{x}}{\sqrt{\bar{a}}}\right)+\bar{r}_0\sqrt{\bar{a}}\sin\left(\frac{\bar{x}}{\sqrt{\bar{a}}}\right)-\sqrt{\mu}t=0$$

$$\eta_2\left(x,t,\Delta\mathbf{v}_0\right)=a\left(x-\sqrt{a}\sin\frac{x}{\sqrt{a}}\right)+a\frac{\mathbf{r}_0^T\left(\mathbf{v}_0+\Delta\mathbf{v}_0\right)}{\sqrt{\mu}}\left(1-\cos\frac{x}{\sqrt{a}}\right)+r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right)-\sqrt{\mu}t=0$$

$$R\left(\bar{x},x,t,\Delta\mathbf{v}_0\right)=\bar{\mathbf{r}}-\mathbf{r}=0$$

$$h=\Delta\mathbf{v}_0^T L_0\left[\left(\frac{\partial\bar{\mathbf{r}}}{\partial\bar{x}}-\frac{\partial\mathbf{r}}{\partial x}\right)+\frac{\bar{r}}{\sqrt{\mu}}\left(\bar{\mathbf{v}}_0-\left(\mathbf{v}_0+\Delta\mathbf{v}_0\right)\right)\right]=0$$

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Spacecraft Interceptor Problem #2(1/7)

Likewise,

- Performance Index

$$J = \frac{1}{2}\Delta\mathbf{v}_1^T \Delta\mathbf{v}_1$$

- Constraints

$$\boldsymbol{\eta}\left(\bar{x}, x_1, x, t, t_1, \Delta v_0\right) = 0$$
$$\bar{\mathbf{r}} - \mathbf{r} = 0$$

Where,

$$\boldsymbol{\eta} = \begin{bmatrix}\eta_1 & \eta_2 & \eta_3\end{bmatrix}^T \in R^3$$

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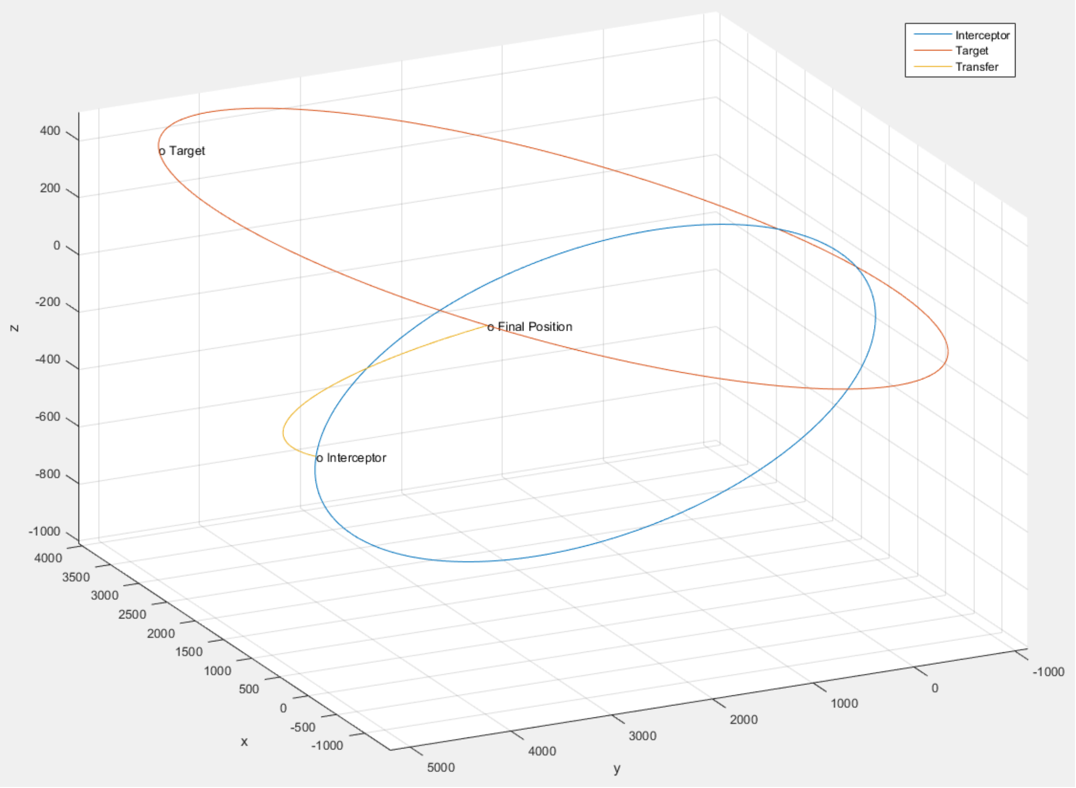
Problem #1 - Result

Initial Condition	Position(km)	Velocity(km/s)
Target	$[3000 \quad 5000 \quad 500]^T$	$[-4.0 \quad 2.0 \quad 0.0]^T$
Interceptor	$[2000 \quad 4000 \quad -500]^T$	$[-3.0 \quad 1.0 \quad 2.0]^T$



Variable	Value
$\bar{x}$	80.2407
x	88.5038
t	642.3342
Δv_{0x}	0.6884
Δv_{0y}	3.9891
Δv_{0z}	-1.1219

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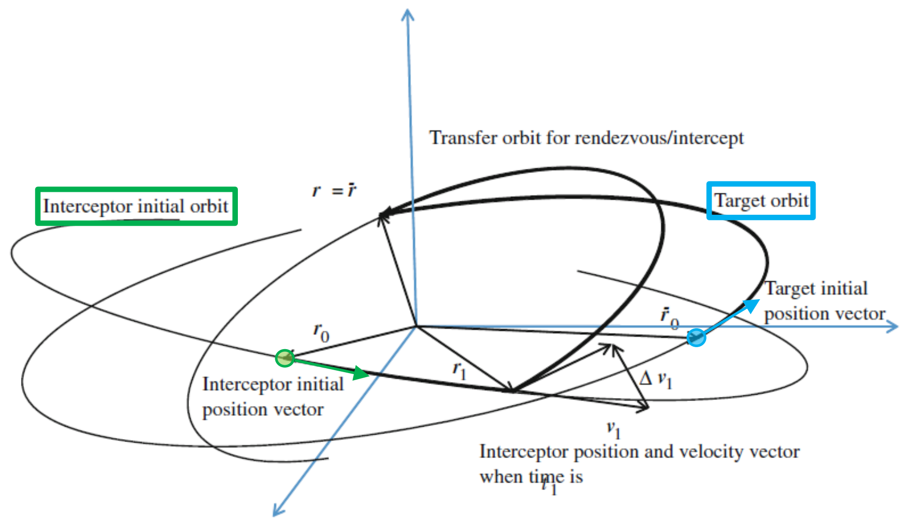
- Spacecraft Interceptor Problem #2 -

ALGORITHM

Spacecraft Interceptor Problem #2(2/7)



- Target Orbit

$$\eta_1(\bar{x},t) = \bar{a}\left(\bar{x} - \sqrt{\bar{a}}\sin\frac{\bar{x}}{\sqrt{\bar{a}}}\right) + \bar{a}\frac{\bar{\mathbf{r}}_0^T\bar{\mathbf{v}}_0}{\sqrt{\mu}}\left(1-\cos\frac{\bar{x}}{\sqrt{\bar{a}}}\right) + \bar{r}_0\sqrt{\bar{a}}\sin\left(\frac{\bar{x}}{\sqrt{\bar{a}}}\right) - \sqrt{\mu}t = 0$$

- Interceptor Orbit – Before the Wait Time

$$\eta_2(x,\Delta v_0,t) = a\left(x - \sqrt{a}\sin\frac{x}{\sqrt{a}}\right) + a\frac{\mathbf{r}_0^T(\mathbf{v}_0+\Delta\mathbf{v}_0)}{\sqrt{\mu}}\left(1-\cos\frac{x}{\sqrt{a}}\right) + r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right) - \sqrt{\mu}t = 0$$

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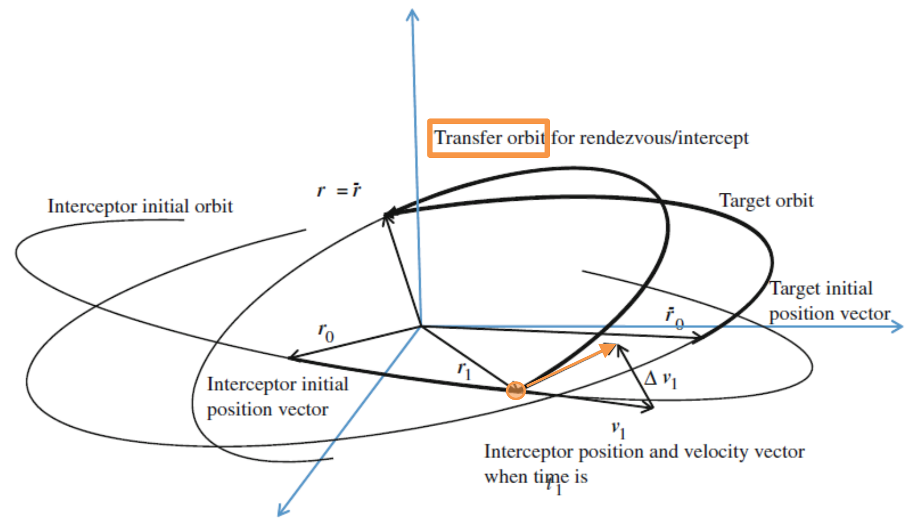
Spacecraft Interceptor Problem #2(3/7)



- Transfer Orbit(Interceptor Orbit - After the Wait Time)

$$\eta_3(x_1, x, t, t_1, \Delta\mathbf{v}_1) = a_1\left(x - \sqrt{a_1}\sin\frac{x}{\sqrt{a_1}}\right) + a_1\frac{\mathbf{r}_1^T(\mathbf{v}_1 + \Delta\mathbf{v}_1)}{\sqrt{\mu}}\left(1 - \cos\frac{x}{\sqrt{a_1}}\right)$$

$$+ r_1\sqrt{a_1}\sin\left(\frac{x}{\sqrt{a_1}}\right) - \sqrt{\mu}(t - t_1) = 0$$

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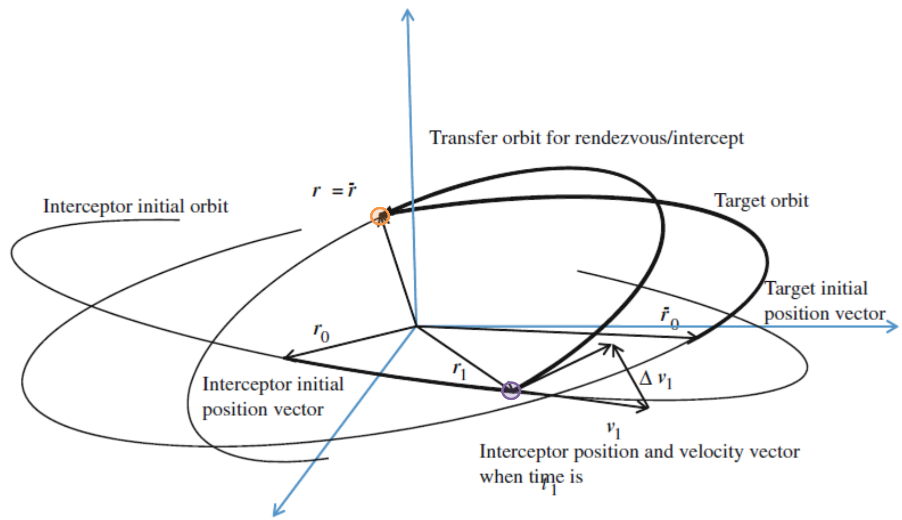
Spacecraft Interceptor Problem #2(4/7)



- The Position and Velocity Vectors of Interceptor at the Wait Time

$$\mathbf{r}_1(x_1,t_1) = f_0\mathbf{r}_0 + g_0\mathbf{v}_0$$
$$\mathbf{v}_1(x_1) = \dot{f}_0\mathbf{r}_0 + \dot{g}_0\mathbf{v}_0$$

- The Final Position and Velocity Vectors of Interceptor

$$\mathbf{r}(x,x_1,\Delta\mathbf{v}_1,t,t_1) = f_1\mathbf{r}_1 + g_1(\mathbf{v}_1+\Delta\mathbf{v}_1)$$
$$\mathbf{v}(x,x_1,\Delta\mathbf{v}_1) = \dot{f}_1\mathbf{r}_1 + \dot{g}_1(\mathbf{v}_1+\Delta\mathbf{v}_1)$$

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Spacecraft Interceptor Problem #2(5/7)

- Cost Function

$$\mathbf{H} = J\left(\Delta\mathbf{v}_1\right) + \boldsymbol{\lambda}^T \boldsymbol{\eta}\left(\bar{x}, x_1, x, t, t_1, \Delta\mathbf{v}_1\right) + \Phi^T\left(\bar{\mathbf{r}} - \mathbf{r}\right)$$

- The Necessary Condition

$$\frac{\partial H}{\partial \bar{x}} = \lambda_1 \frac{\partial \eta_1}{\partial \bar{x}} + \Phi^T \frac{\partial \bar{\mathbf{r}}}{\partial \bar{x}} = 0$$

$$\frac{\partial H}{\partial x_1} = \lambda_2 \frac{\partial \eta_2}{\partial x_1} + \lambda_3 \frac{\partial \eta_3}{\partial x_1} - \Phi^T \frac{\partial \mathbf{r}}{\partial x_1} = 0$$

$$\frac{\partial H}{\partial x} = \lambda_3 \frac{\partial \eta_3}{\partial x} - \Phi^T \frac{\partial \mathbf{r}}{\partial \bar{x}} = 0$$

$$\frac{\partial H}{\partial t} = -\sqrt{\mu}\left(\lambda_1 + \lambda_3\right) + \Phi^T\left(\frac{\partial \bar{\mathbf{r}}}{\partial t} - \frac{\partial \mathbf{r}}{\partial t}\right) = 0$$

$$\frac{\partial H}{\partial t_1} = -\sqrt{\mu}\lambda_2 + \lambda_3 \frac{\partial \eta_3}{\partial t_1} - \Phi^T \frac{\partial \mathbf{r}}{\partial t_1} = 0$$

$$\frac{\partial H}{\partial \Delta\mathbf{v}_1} = \frac{\partial J}{\partial \Delta\mathbf{v}_1} + \lambda_3 \frac{\partial \eta_3}{\partial \Delta\mathbf{v}_1} - \Phi^T \frac{\partial \mathbf{r}}{\partial \Delta\mathbf{v}_1} = 0$$

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Spacecraft Interceptor Problem #2(6/7)

- Lagrange Multipliers

$$\lambda_1 = -\Phi^T \frac{1}{\bar{r}} \frac{\partial \bar{\mathbf{r}}}{\partial \bar{x}} \quad / \quad \lambda_2 = -\Phi^T \frac{1}{r} \frac{1}{r_1} \frac{\partial \eta_3}{\partial x_1} \frac{\partial \mathbf{r}}{\partial x} + \Phi^T \frac{1}{r_1} \frac{\partial \mathbf{r}}{\partial x_1} \quad / \quad \lambda_3 = \Phi^T \frac{1}{r} \frac{\partial \mathbf{r}}{\partial x}$$

$$\Phi^T = \frac{\partial J}{\partial \Delta \mathbf{v}_1} \left(\frac{\partial \mathbf{r}}{\partial \Delta \mathbf{v}_1} - \frac{1}{r} \frac{\partial \mathbf{r}}{\partial x} \frac{\partial \eta_3}{\partial \Delta \mathbf{v}_1} \right)^{-1}$$

- The interceptor condition is rewritten as

$$h_1 = \Delta \mathbf{v}_1^T L_1 \left[\left(\frac{\partial \bar{\mathbf{r}}}{\partial \bar{x}} - \frac{\partial \mathbf{r}}{\partial x} \right) + \frac{r}{\sqrt{\mu}} \left(\bar{\mathbf{v}}_0 - \left(\mathbf{v}_1 + \Delta \mathbf{v}_1 \right) \right) \right] = 0$$

$$h_2 = \Delta \mathbf{v}_1^T L_1 \left[\frac{1}{r} \left(\frac{\sqrt{\mu}}{r_1} \frac{\partial \eta_3}{\partial x_1} + \frac{\partial \eta_3}{\partial t_1} \right) \frac{\partial \mathbf{r}}{\partial x} - \left(\frac{\sqrt{\mu}}{r_1} \frac{\partial \mathbf{r}}{\partial x_1} + \frac{\partial \mathbf{r}}{\partial x_1} \right) \right] = 0$$

Where,

$$L_1 = \left(\frac{\partial \mathbf{r}}{\partial \Delta \mathbf{v}_1} - \frac{1}{r} \frac{\partial \mathbf{r}}{\partial x} \frac{\partial \eta_3}{\partial \Delta \mathbf{v}_1} \right)^{-1}$$

ALGORITHM

Spacecraft Interceptor Problem #2(7/7)

Finally,

$$\eta_1(\bar{x},t)=\bar{a}\left(\bar{x}-\sqrt{\bar{a}}\sin\frac{\bar{x}}{\sqrt{\bar{a}}}\right)+\bar{a}\frac{\bar{\mathbf{r}}_0^T\bar{\mathbf{v}}_0}{\sqrt{\mu}}\left(1-\cos\frac{\bar{x}}{\sqrt{\bar{a}}}\right)+\bar{r}_0\sqrt{\bar{a}}\sin\left(\frac{\bar{x}}{\sqrt{\bar{a}}}\right)-\sqrt{\mu}t=0$$

$$\eta_2(x,\Delta v_0,t)=a\left(x-\sqrt{a}\sin\frac{x}{\sqrt{a}}\right)+a\frac{\mathbf{r}_0^T(\mathbf{v}_0+\Delta\mathbf{v}_0)}{\sqrt{\mu}}\left(1-\cos\frac{x}{\sqrt{a}}\right)+r_0\sqrt{a}\sin\left(\frac{x}{\sqrt{a}}\right)-\sqrt{\mu}t=0$$

$$\eta_3(x_1,x,t,t_1,\Delta\mathbf{v}_1)=a_1\left(x-\sqrt{a_1}\sin\frac{x}{\sqrt{a_1}}\right)+a_1\frac{\mathbf{r}_1^T(\mathbf{v}_1+\Delta\mathbf{v}_1)}{\sqrt{\mu}}\left(1-\cos\frac{x}{\sqrt{a_1}}\right)$$

$$+r_1\sqrt{a_1}\sin\left(\frac{x}{\sqrt{a_1}}\right)-\sqrt{\mu}(t-t_1)=0$$

$$R(\bar{x},x,x_1,t,t_1,\Delta\mathbf{v}_1)=\bar{\mathbf{r}}-\mathbf{r}=0$$

$$h_1=\Delta\mathbf{v}_1^T L_1\left[\left(\frac{\partial\bar{\mathbf{r}}}{\partial\bar{x}}-\frac{\partial\mathbf{r}}{\partial x}\right)+\frac{r}{\sqrt{\mu}}\left(\bar{\mathbf{v}}_0-(\mathbf{v}_1+\Delta\mathbf{v}_1)\right)\right]=0$$

$$h_2=\Delta\mathbf{v}_1^T L_1\left[\frac{1}{r}\left(\frac{\sqrt{\mu}}{r_1}\frac{\partial\eta_3}{\partial x_1}+\frac{\partial\eta_3}{\partial t_1}\right)\frac{\partial\mathbf{r}}{\partial x}-\left(\frac{\sqrt{\mu}}{r_1}\frac{\partial\mathbf{r}}{\partial x_1}+\frac{\partial\mathbf{r}}{\partial x_1}\right)\right]=0$$

THANK YOU

- Question & Answer -