



Two-Impulse Rendezvous Maneuvers of Spacecraft in relative motion

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1 Relative motion in orbit



$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = (r_A v_{A\perp}) \hat{\mathbf{k}} = (r_A^2 \Omega) \hat{\mathbf{k}} = r_A^2 \Omega$$

$$\Omega = \frac{\mathbf{h}_A}{r_A^2} = \frac{\mathbf{r}_A \times \mathbf{v}_A}{r_A^2} \quad \dot{\Omega} = \mathbf{h}_A \frac{d}{dt} \left(\frac{1}{r_A^2} \right) = -2 \frac{\mathbf{h}_A}{r_A^3} \dot{r}_A$$

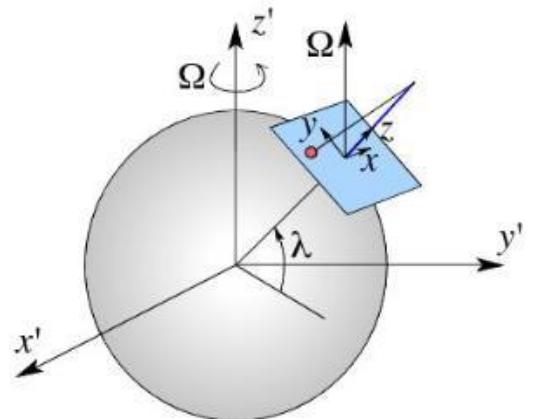
$$(\dot{r}_A = \mathbf{v}_A \cdot \mathbf{r}_A / r_A) \quad \dot{\Omega} = -2 \frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^4} \mathbf{h}_A = -2 \frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^2} \Omega$$

[Geocentric Frame]

$$\mathbf{r}_{rel} = \mathbf{r}_B - \mathbf{r}_A$$

$$\mathbf{v}_{rel} = \mathbf{v}_B - \mathbf{v}_A - \dot{\Omega} \times \mathbf{r}_{rel}$$

$$\mathbf{a}_{rel} = \mathbf{a}_B - \mathbf{a}_A - \dot{\Omega} \times (\dot{\Omega} \times \mathbf{r}_{rel}) - 2\dot{\Omega} \times \mathbf{v}_{rel}$$





1 Relative motion in orbit

$$\begin{aligned}\{\mathbf{r}_{rel}\}_X &= \begin{Bmatrix} X_B - X_A \\ Y_B - Y_A \\ Z_B - Z_A \end{Bmatrix} \\ \{\mathbf{v}_{rel}\}_X &= \begin{Bmatrix} \dot{X}_B - \dot{X}_A + \Omega_Z(Y_B - Y_A) - \Omega_Y(Z_B - Z_A) \\ \dot{Y}_B - \dot{Y}_A + \Omega_Z(X_B - X_A) - \Omega_X(Z_B - Z_A) \\ \dot{Z}_B - \dot{Z}_A + \Omega_Y(X_B - X_A) - \Omega_X(Y_B - Y_A) \end{Bmatrix} \\ \{\mathbf{a}_{rel}\}_X &= \begin{Bmatrix} \ddot{X}_B - \ddot{X}_A + 2\Omega_Z(\dot{Y}_B - \dot{Y}_A) - 2\Omega_Y(\dot{Z}_B - \dot{Z}_A) - (\Omega_Y^2 + \Omega_Z^2)(X_B - X_A) \\ + (\Omega_X\Omega_Y + a\Omega_Z)(Y_B - Y_A) + (\Omega_X\Omega_Z - a\Omega_Y)(Z_B - Z_A) \\ \ddot{Y}_B - \ddot{Y}_A + 2\Omega_Z(\dot{X}_B - \dot{X}_A) - 2\Omega_X(\dot{Z}_B - \dot{Z}_A) + (\Omega_X^2 + \Omega_Z^2)(Y_B - Y_A) \\ + (\Omega_X\Omega_Y - a\Omega_Z)(X_B - X_A) + (\Omega_Y\Omega_Z + a\Omega_X)(Z_B - Z_A) \\ \ddot{Z}_B - \ddot{Z}_A + 2\Omega_Y(\dot{X}_B - \dot{X}_A) - 2\Omega_Y(\dot{Y}_B - \dot{Y}_A) + (\Omega_X^2 - \Omega_Y^2)(Z_B - Z_A) \\ + (\Omega_X\Omega_Z + a\Omega_Y)(X_B - X_A) + (\Omega_Y\Omega_Z - a\Omega_X)(Y_B - Y_A) \end{Bmatrix}\end{aligned}$$



1 Relative motion in orbit

$$\hat{\mathbf{i}} = l_x \hat{\mathbf{I}} + m_x \hat{\mathbf{J}} + n_x \hat{\mathbf{K}}$$

$$\hat{\mathbf{j}} = l_y \hat{\mathbf{I}} + m_y \hat{\mathbf{J}} + n_y \hat{\mathbf{K}}$$

$$\hat{\mathbf{k}} = l_z \hat{\mathbf{I}} + m_z \hat{\mathbf{J}} + n_z \hat{\mathbf{K}}$$

$$[\mathbf{Q}]_{Xx} = \begin{matrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{matrix}$$

$$\{\mathbf{r}_{rel}\}_x = [\mathbf{Q}]_{Xx} \{\mathbf{r}_{rel}\}_X$$

$$\{\mathbf{v}_{rel}\}_x = [\mathbf{Q}]_{Xx} \{\mathbf{v}_{rel}\}_X$$

$$\{\mathbf{a}_{rel}\}_x = [\mathbf{Q}]_{Xx} \{\mathbf{a}_{rel}\}_X$$

1 Relative motion in orbit

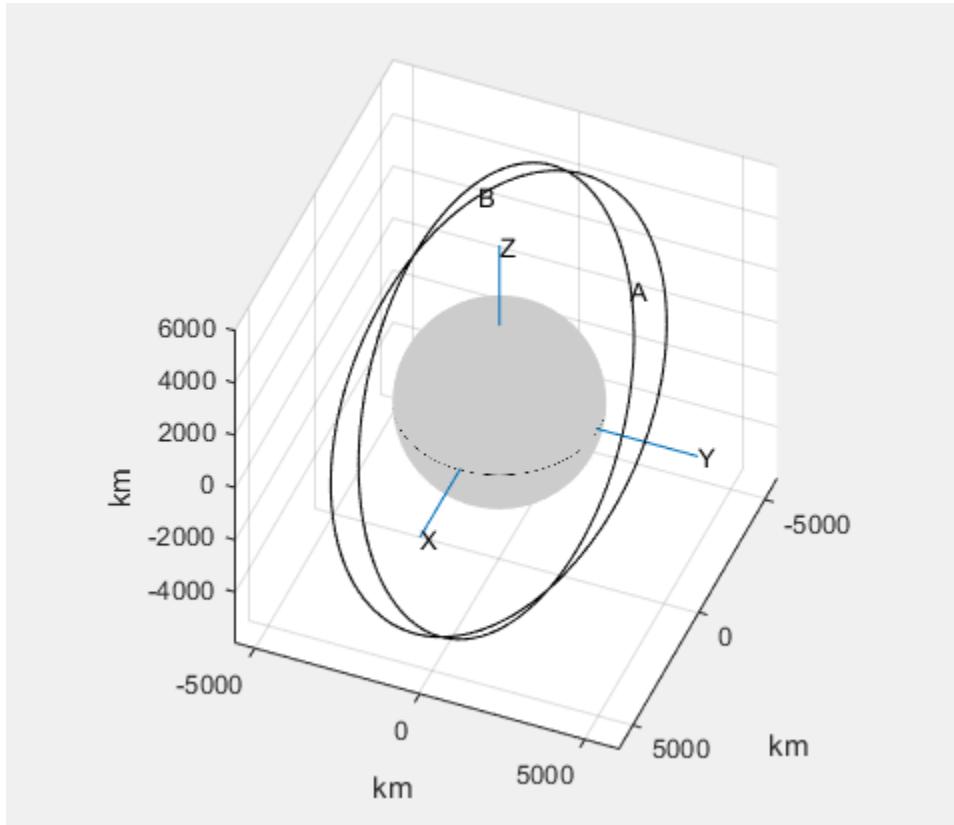


Fig.1.1) Spacecraft *A* and *B*'s orbits using $\{\mathbf{r}_{rel}\}_X$

1 Relative motion in orbit

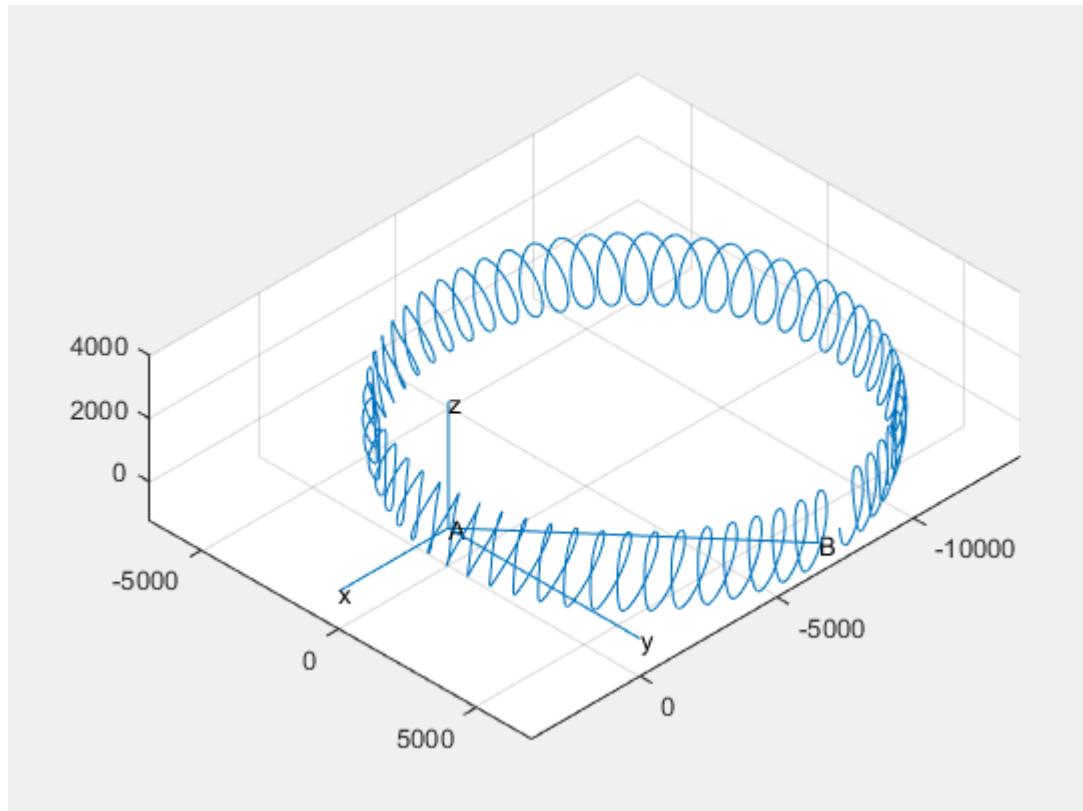


Fig.1.2) Trajectory of spacecraft *B* relative to *A*



2 Linearization of relative motion eq.

$$\ddot{\delta \mathbf{r}} = -\ddot{\mathbf{R}} - \mu \frac{\mathbf{R} + \delta \mathbf{r}}{r^3} \quad \text{----- (1)}$$

Non-Linearization

$$\mathbf{r} = \mathbf{R} + \delta \mathbf{r}$$

$$r^2 = \mathbf{r} \cdot \mathbf{r} = (\mathbf{R} + \delta \mathbf{r}) \cdot (\mathbf{R} + \delta \mathbf{r}) = \mathbf{R} \cdot \mathbf{R} + 2\mathbf{R} \cdot \delta \mathbf{r} + \delta \mathbf{r} \cdot \delta \mathbf{r}$$

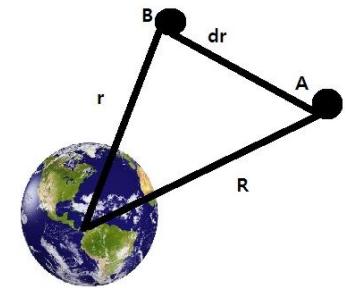
$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5} \mathbf{R} \cdot \delta \mathbf{r} \quad \text{----- (2)}$$

$$\ddot{\delta \mathbf{r}} = -\ddot{\mathbf{R}} - \mu \left(\frac{1}{R^3} - \frac{3}{R^5} \mathbf{R} \cdot \delta \mathbf{r} \right) (\mathbf{R} + \delta \mathbf{r}) \quad \text{----- (3)}$$

$$= -\ddot{\mathbf{R}} - \mu \left[\frac{\mathbf{R}}{R^3} + \frac{\delta \mathbf{r}}{R^3} - \frac{3}{R^5} (\mathbf{R} \cdot \delta \mathbf{r}) \mathbf{R} + (\delta \mathbf{r} \ll 0) \right]$$

$$\ddot{\mathbf{R}} = -\mu \frac{\mathbf{R}}{R^3} \quad \text{----- (4)}$$

$$\ddot{\delta \mathbf{r}} = -\frac{\mu}{R^3} \left[\delta \mathbf{r} - \frac{3}{R^2} (\mathbf{R} \cdot \delta \mathbf{r}) \mathbf{R} \right] \quad \text{----- (5)} \quad \text{Linearization !!}$$





2 Linearization of relative motion eq.

$$\delta \mathbf{r} = \delta x \hat{\mathbf{i}} + \delta y \hat{\mathbf{j}} + \delta z \hat{\mathbf{k}}$$

$$\ddot{\delta \mathbf{r}} = -\frac{\mu}{R^3} \left[(\delta x \hat{\mathbf{i}} + \delta y \hat{\mathbf{j}} + \delta z \hat{\mathbf{k}}) - \frac{3}{R^2} \left[(\mathbf{R} \hat{\mathbf{i}}) \cdot (\delta x \hat{\mathbf{i}} + \delta y \hat{\mathbf{j}} + \delta z \hat{\mathbf{k}}) \right] (\mathbf{R} \hat{\mathbf{i}}) \right]$$

$$\ddot{\delta \mathbf{a}_{rel}} = \ddot{\delta \mathbf{r}} - \dot{\Omega} \times \delta \mathbf{r} - \Omega \times (\Omega \times \delta \mathbf{r}) - 2\Omega \times \delta \mathbf{v}_{rel}$$

$$\ddot{\delta \mathbf{a}_{rel}} = -\frac{\mu}{R^3} (-2\delta x \hat{\mathbf{i}} + \delta y \hat{\mathbf{j}} + \delta z \hat{\mathbf{k}}) - \frac{2(\mathbf{V} \cdot \mathbf{R})h}{R^4} (\delta y \hat{\mathbf{i}} - \delta x \hat{\mathbf{j}}) - \left[-\frac{h^2}{R^4} (\delta x \hat{\mathbf{i}} + \delta y \hat{\mathbf{j}}) \right] - 2 \frac{h}{R^2} (\delta x \hat{\mathbf{j}} + \delta y \hat{\mathbf{i}})$$

$$\ddot{\delta \mathbf{a}_{rel}} = \ddot{\delta x} \hat{\mathbf{i}} + \ddot{\delta y} \hat{\mathbf{j}} + \ddot{\delta z} \hat{\mathbf{k}}$$

$$\ddot{\delta x} - \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4} \right) \delta x + \frac{2(\mathbf{V} \cdot \mathbf{R})h}{R^4} \delta y - 2 \frac{h}{R^2} \dot{\delta y} = 0$$

$$\ddot{\delta y} + \left(\frac{\mu}{R^3} - \frac{h^2}{R^4} \right) \delta y - \frac{2(\mathbf{V} \cdot \mathbf{R})h}{R^4} \delta x + 2 \frac{h}{R^2} \dot{\delta x} = 0$$

$$\ddot{\delta z} + \frac{\mu}{R^3} \delta z = 0$$

2 Linearization of relative motion eq.

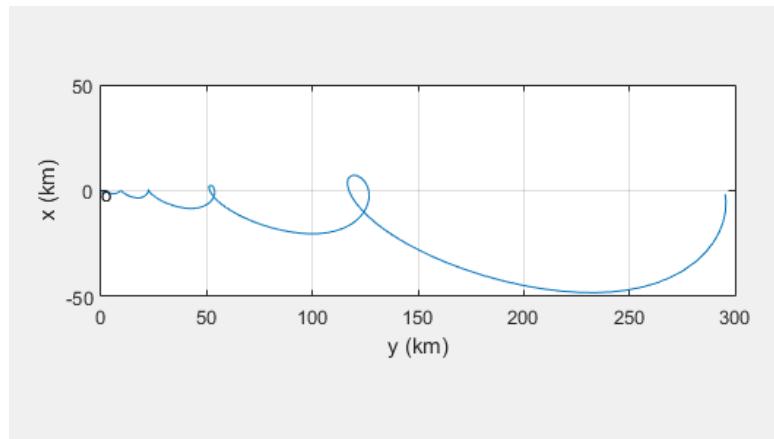


Fig.2.1) Trajectory of B relative to A in the co-moving frame ($e=0.1$)

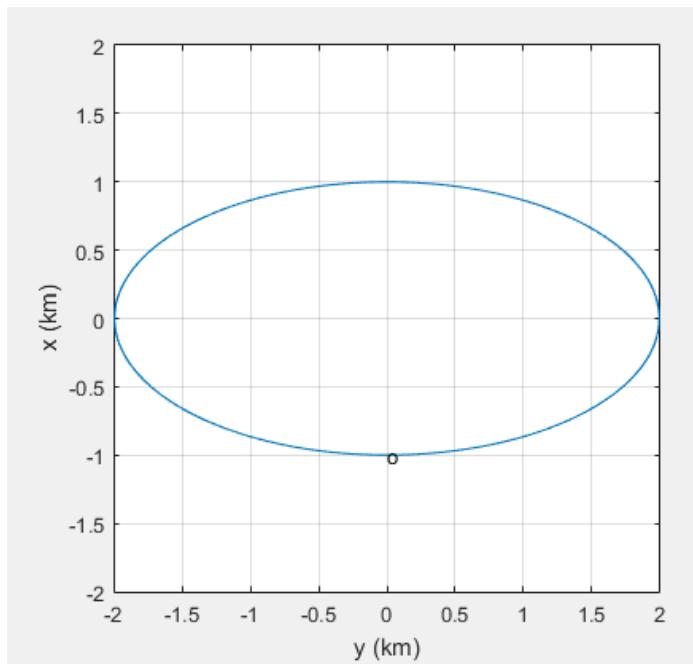


Fig.2.2) Trajectory of B relative to A in the co-moving frame ($e=0$)



3 Clohessy-wiltshire equations

$$\delta \ddot{x} - 3 \frac{\mu}{R^3} \delta x - 2 \sqrt{\frac{\mu}{R^3}} \delta \dot{y} = 0$$

$$\delta \ddot{y} + 2 \sqrt{\frac{\mu}{R^3}} \delta \dot{x} = 0$$

$$\delta \ddot{z} + \frac{\mu}{R^3} \delta z = 0$$

$$n = \frac{V}{R} = \frac{\sqrt{\mu / R}}{R} = \sqrt{\frac{\mu}{R^3}}$$

$$\delta \ddot{x} - 3n^2 \delta x - 2n \delta \dot{y} = 0 \quad ---(1)$$

$$\delta \ddot{y} + 2n \delta \dot{x} = 0 \quad ---(2)$$

$$\delta \ddot{z} + n^2 \delta z = 0 \quad ---(3)$$

$$\delta \dot{y} = C_1 - 2n \delta x \quad ---(4)$$

$$\delta \ddot{x} + n^2 \delta x = 2nC_1 \quad ---(5)$$

$$\delta x = \frac{2}{n} C_1 + C_2 \sin nt + C_3 \cos nt \quad ---(6)$$

$$\delta \dot{x} = C_2 \cos nt - C_3 \sin nt \quad ---(7)$$

$$\delta \dot{y} = -3C_1 - 2C_2 n \sin nt - 2C_3 n \cos nt \quad ---(8)$$

$$\delta y = -3C_1 t + 2C_2 \cos nt - 2C_3 \sin nt + C_4 \quad ---(9)$$

$$t = 0$$

$$\delta x = \delta x_0$$

$$\delta y = \delta y_0$$

$$\delta \dot{x} = \delta \dot{x}_0$$

$$\delta \dot{y} = \delta \dot{y}_0$$

$$\frac{2}{n} C_1 + C_3 = \delta x_0$$

$$C_2 n = \delta \dot{x}_0$$

$$-3C_1 - 2C_3 n = \delta \dot{y}_0$$

$$2C_2 + C_4 = \delta y_0$$

$$C_1 = 2n \delta x_0 + \delta \dot{y}_0$$

$$C_2 = \frac{1}{n} \delta \dot{x}_0$$



3 Clohessy-wiltshire equations

$$C_3 = -3\delta x_0 - \frac{2}{n}\delta \dot{y}_0$$

$$C_4 = -\frac{2}{n}\delta \dot{x}_0 + \delta y_0$$

$$\delta z = C_5 \sin nt + C_6 \cos nt$$

$$\delta \dot{z} = C_5 n \cos nt - C_6 n \sin nt$$

$$C_5 = \frac{\delta \dot{z}_0}{n}$$

$$C_6 = \delta z_0$$

$$\delta x = 4\delta x_0 + \frac{2}{n}\delta \dot{y}_0 + \frac{\delta \dot{x}_0}{n} \sin nt - \left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0 \right) \cos nt$$

$$\delta y = \delta y_0 - \frac{2}{n}\delta \dot{x}_0 - 3(2n\delta x_0 + \delta \dot{y}_0)t + 2\left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0 \right) \sin nt + \frac{2}{n}\delta \dot{x}_0 \cos nt$$

$$\delta z = \frac{1}{n}\delta \dot{z}_0 \sin nt + \delta z_0 \cos nt$$



3 Clohessy-wiltshire equations

$$\delta u = \delta \dot{x}$$

$$\delta v = \delta \dot{y}$$

$$\delta w = \delta \dot{z}$$

$$\delta u_0 = \delta \dot{x}_0$$

$$\delta v_0 = \delta \dot{y}_0$$

$$\delta w = \delta \dot{z}_0$$

$$\delta x = (4 - 3 \cos nt) \delta x_0 + \frac{\sin nt}{n} \delta u_0 + \frac{2}{n} (1 - \cos nt) \delta v_0$$

$$\delta y = 6(\sin nt - nt) \delta x_0 + \delta y_0 + \frac{2}{n} (\cos nt - 1) \delta u_0 + \frac{1}{n} (4 \sin nt - 3nt) \delta v_0$$

$$\delta z = \cos nt \delta z_0 + \frac{1}{n} \sin nt \delta w_0$$

$$\delta u = 3n \sin nt \delta x_0 + \cos nt \delta u_0 + 2 \sin nt \delta v_0$$

$$\delta v = 6n (\cos nt - 1) \delta x_0 - 2 \sin nt \delta u_0 + (3 \cos nt - 3) \delta v_0$$

$$\delta w = -n \sin nt \delta z_0 + \cos nt \delta w_0$$



3 Clohessy-wiltshire equations

$$\{\delta \mathbf{r}(t)\} = [\Phi_{rr}(t)]\{\delta \mathbf{r}_0\} + [\Phi_{rv}(t)]\{\delta \mathbf{v}_0\}$$

$$\{\delta \mathbf{v}(t)\} = [\Phi_{vr}(t)]\{\delta \mathbf{r}_0\} + [\Phi_{vv}(t)]\{\delta \mathbf{v}_0\}$$

$$[\Phi_{rr}(t)] = \begin{bmatrix} 4 - 3 \cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix}$$

$$[\Phi_{rv}(t)] = \begin{bmatrix} \frac{1}{n} \sin nt & \frac{2}{n}(1 - \cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4 \sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n} \sin nt \end{bmatrix}$$



3 Clohessy-wiltshire equations

$$[\Phi_{vr}(t)] = \begin{bmatrix} 3n \sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n \sin nt \end{bmatrix}$$

$$[\Phi_{vv}(t)] = \begin{bmatrix} \cos nt & 2 \sin nt & 0 \\ -2 \sin nt & 4 \cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix}$$



4 Two-impulse rendezvous maneuvers

$$\Delta \mathbf{v}_0 = \delta \mathbf{v}_0^+ - \delta \mathbf{v}_0^- = (\delta u_0^+ - \delta u_0^-) \hat{\mathbf{i}} + (\delta v_0^+ - \delta v_0^-) \hat{\mathbf{j}} + (\delta w_0^+ - \delta w_0^-) \hat{\mathbf{k}}$$

$$\{0\} = [\Phi_{rr}(t_f)]\{\delta \mathbf{r}_0\} + [\Phi_{rv}(t_f)]\{\delta \mathbf{v}_0^+\}$$

$$\{\delta \mathbf{v}_0^+\} = -[\Phi_{rv}(t_f)]^{-1} [\Phi_{rr}(t_f)]\{\delta \mathbf{r}_0\}$$

$$\begin{aligned}\{\delta \mathbf{v}_f^-\} &= [\Phi_{vr}(t_f)]\{\delta \mathbf{r}_0\} + [\Phi_{vv}(t_f)]\{\delta \mathbf{v}_0\} \\ &= [\Phi_{vr}(t_f)]\{\delta \mathbf{r}_0\} + [\Phi_{vv}(t_f)](-[\Phi_{rv}(t_f)]^{-1} [\Phi_{rr}(t_f)]\{\delta \mathbf{r}_0\})\end{aligned}$$

$$\{\delta \mathbf{v}_f^-\} = ([\Phi_{vr}(t_f)] - [\Phi_{vv}(t_f)]^{-1} [\Phi_{rr}(t_f)])\{\delta \mathbf{r}_0\} \quad \Delta \mathbf{v}_f = \delta \mathbf{v}_f^+ - \delta \mathbf{v}_f^- = 0 - \delta \mathbf{v}_f^- = -\delta \mathbf{v}_f^-$$



4 Two-impulse rendezvous maneuvers

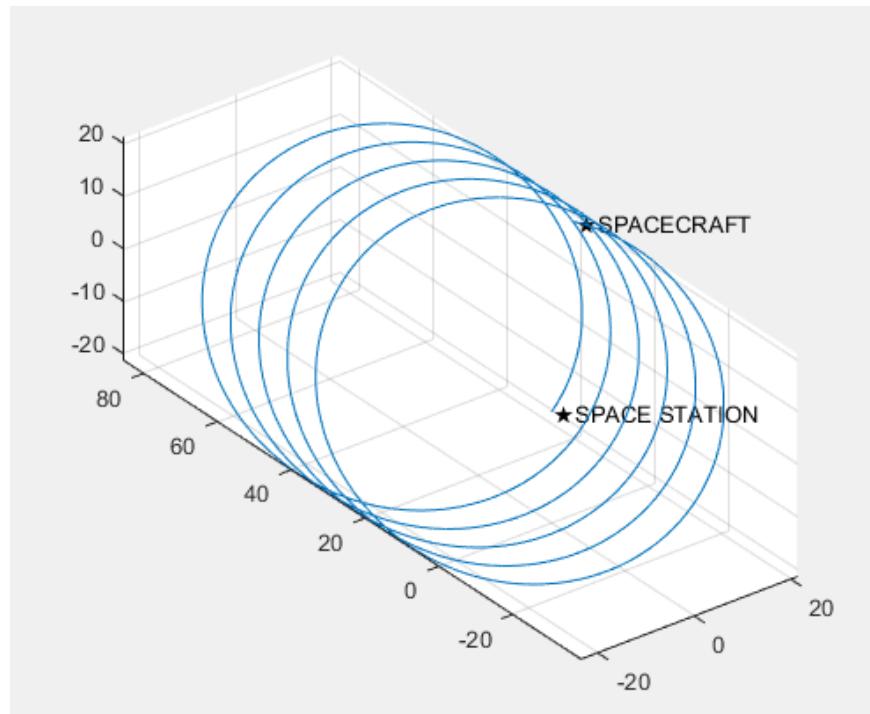


Fig.4.1) Rendezvous trajectory of the chaser vehicle relative to the target

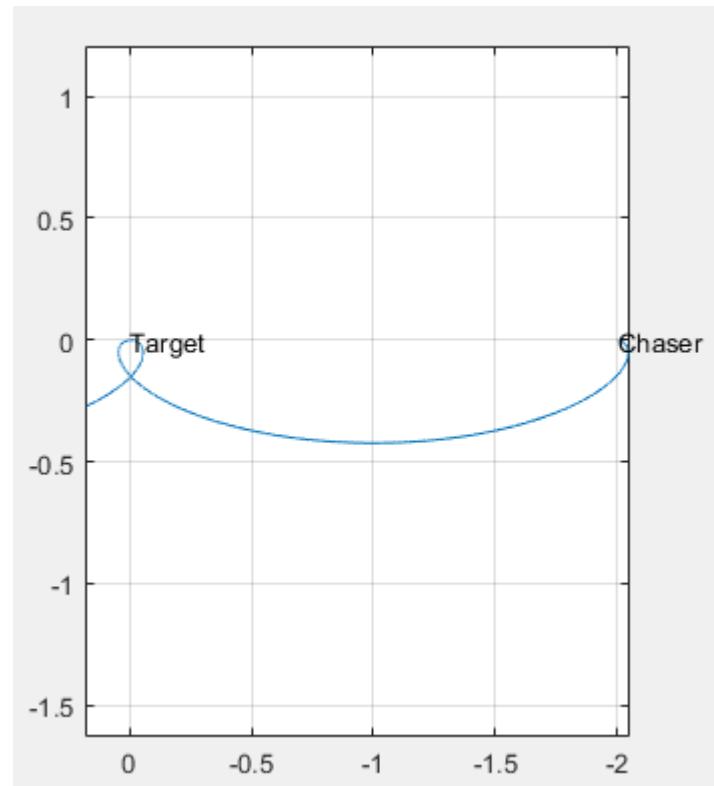


Fig.4.2) Motion of the chaser relative to the target



Thank you